Every compact convex subset of matrices is the Clarke Jacobian of some Lipschitzian mapping

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Outline of the talk

• Introduction: basic concepts, the question and our answer

• Five lemmas: ray-fish, ray-fish colony, ray-fish colony for a line segment,
  ray-fish colony for a polygonal chain, and corona

• Finale: the main result and its proof
Introduction
Introduction: Basic concepts

Let \( f: \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a Lipschitzian mapping; that is, a mapping such that
\[
\|f(x) - f(y)\| \leq L \|x - y\| \quad \text{for every } x, y \in \mathbb{R}^n
\]
for some constant \( L > 0 \), where the norms \( \|\cdot\| \) are Euclidean.

The Euclidean vector space \( \mathbb{R}^n \) and \( \mathbb{R}^m \) of dimension \( n \) and \( m \), respectively, is identified with the space \( \mathbb{R}^{n\times 1} \) and \( \mathbb{R}^{m\times 1} \), respectively, by convention. Therefore, a vector \( x \in \mathbb{R}^n \) and \( z \in \mathbb{R}^m \) is understood as column vector of \( n \) and \( m \), respectively, real numbers.
Introduction: Basic concepts

Every matrix $A \in \mathbb{R}^{m \times n}$ induces a linear mapping $L_A: \mathbb{R}^n \rightarrow \mathbb{R}^m$ by

$$L_A: x \mapsto Ax \quad \text{for} \quad x \in \mathbb{R}^n$$

and every linear mapping $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is induced by a matrix $A_L \in \mathbb{R}^{m \times n}$ so that

$$L(x) = A_L x \quad \text{for} \quad x \in \mathbb{R}^n$$

This is the reason why we shall identify each matrix $A \in \mathbb{R}^{m \times n}$ with the respective linear mapping which it induces, and vice versa.
Introduction: Basic concepts

By Rademacher’s Theorem, the given Lipschitzian mapping \( f: \mathbb{R}^n \rightarrow \mathbb{R}^m \) is Gâteaux (therefore: Fréchet) differentiable almost everywhere with respect to the \( n \)-dimensional Lebesgue measure on \( \mathbb{R}^n \).

That is, the Jacobian matrix

\[
J_f(x) = \begin{pmatrix}
\frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_m(x)}{\partial x_1} & \frac{\partial f_m(x)}{\partial x_2} & \cdots & \frac{\partial f_m(x)}{\partial x_n}
\end{pmatrix}
\]

is defined for almost all \( x \in \mathbb{R}^n \).

The Jacobian matrix \( J_f(x) \) is identified with the Gâteaux derivative, which is the linear mapping \( f'(x): \mathbb{R}^n \rightarrow \mathbb{R}^m \).
Introduction: Basic concepts

The **Bouligand Jacobian** of a Lipschitzian mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point $x_0 \in \mathbb{R}^n$ is the set

$$
\partial_B f(x_0) = \left\{ M \in \mathbb{R}^{m \times n} : \exists (x_k)_{k=1}^{\infty} \subset \mathbb{R}^n : \lim_{k \to \infty} x_k = x_0, \right. \\
\left. \text{function } f \text{ is differentiable at each } x_k \right.$$

and $\lim_{k \to \infty} Jf(x_k) = M \}$

Since the set $\partial_B f(x_0)$ is a collection of matrices, it is identified with the collection of the corresponding linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$. 
Introduction: Basic concepts

The **Clarke Jacobian** of a Lipschitzian mapping \( f: \mathbb{R}^n \to \mathbb{R}^m \) at a point \( x_0 \in \mathbb{R}^n \) is the set

\[
\partial f(x_0) = \text{co } \partial B f(x_0)
\]

It is easy to see that the Clarke Jacobian \( \partial f(x_0) \) is:

- non-empty
- compact (that is closed & bounded)
- convex

The Clarke Jacobian \( \partial f(x_0) \) is also seen as a collection of linear mappings from \( \mathbb{R}^n \) to \( \mathbb{R}^m \).
Introduction: The Question

Given a non-empty compact convex set $\mathcal{P} \subset \mathbb{R}^{m \times n}$ of matrices, is this set the Clarke Jacobian of some Lipschitzian mapping $g: \mathbb{R}^n \to \mathbb{R}^m$ at some point $x_0 \in \mathbb{R}^n$?

In other words, characterize those non-empty compact convex sets $\mathcal{P} \subset \mathbb{R}^{m \times n}$ of matrices that are Clarke Jacobians of some Lipschitzian mappings.
Given a non-empty compact convex set $\mathcal{P} \subset \mathbb{R}^{m \times n}$ of matrices, there exists a Lipschitzian mapping $g: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\partial g(0) = \mathcal{P}$$

We actually prove more…
Consider a linear subspace \( \{0\} \not\subset W \subset \mathbb{R}^n \)

In the following, we identify every matrix \( M \in \mathbb{R}^{m \times n} \) and the linear mapping \( L_M: \mathbb{R}^n \to \mathbb{R}^m \), defined by \( L_M: x \mapsto Mx \) for \( x \in \mathbb{R}^n \), which it induces; that is, we use the same symbol “\( M \)” for both the matrix \( M \) and the mapping \( L_M \).

By \( M|_W \) we denote the linear mapping \( M = L_M \) restricted onto the subspace \( W \), that is the mapping

\[
M|_W: W \to \mathbb{R}^m
\]

\[
M|_W: x \mapsto Mx \quad \text{for} \quad x \in W
\]
Consider the linear subspace \( \{0\} \subset W \subset \mathbb{R}^n \).

Consider also the given non-empty compact convex set \( \mathcal{P} \subset \mathbb{R}^{m \times n} \) of matrices.

By \( \mathcal{P}|_W \) we denote the collection of the restricted linear mappings

\[
\mathcal{P}|_W = \{ M|_W : M \in \mathcal{P} \}
\]

where

\[
M|_W : W \to \mathbb{R}^m \\
M|_W : x \mapsto Mx \quad \text{for} \quad x \in W \quad \text{for every} \quad M \in \mathcal{P}
\]
Consider the linear subspace \( \{0\} \not\subseteq W \subset \mathbb{R}^n \).

Consider also a Lipschitzian mapping \( g: \mathbb{R}^n \to \mathbb{R}^m \).

By \( g|_W \) we denote the restriction of \( g \) onto \( W \), that is the mapping

\[
g|_W: W \to \mathbb{R}^m
\]

\[
g|_W: x \mapsto g(x)
\]

Recall that we identify the Clarke Jacobian \( \partial g(0) \subset \mathbb{R}^{m \times n} \) with the respective collection of the linear mappings \( M: \mathbb{R}^n \to \mathbb{R}^m \) for \( M \in \partial g(0) \).

Therefore, we can define the Clarke Jacobian \( \partial g|_W(0) \) accordingly.
Let $\mathcal{P} \subset \mathbb{R}^{m \times n}$ be a non-empty compact convex set of matrices.

Then there exists a Lipschitzian mapping $g : \mathbb{R}^n \to \mathbb{R}^m$, with $g(0) = 0$, such that, for every linear subspace $\{0\} \subsetneq W \subset \mathbb{R}^n$, the Clarke Jacobian

$$\partial g|_W(0) = \mathcal{P}|_W$$
Five Lemmas
The Ray-Fish Construction: Introduction I

We define that a mapping $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is \textbf{finitely piecewise affine} if there are finitely many pairwise disjoint non-empty open sets $\Omega_1, \ldots, \Omega_k \subset \mathbb{R}^n$ such that $\mathbb{R}^n \setminus \bigcup_{i=1}^k \Omega_i$ is Lebesgue negligible and there are matrices $M_1, \ldots, M_k \in \mathbb{R}^{m \times n}$ and constant vectors $c_1, \ldots, c_k \in \mathbb{R}^m$ such that

$$f(x) = \begin{cases} M_1 x + c_1, & \text{if } x \in \Omega_1 \\ M_2 x + c_2, & \text{if } x \in \Omega_2 \\ \vdots & \vdots \\ M_k x + c_k, & \text{if } x \in \Omega_k \end{cases}$$

Observe that this mapping $f$ is also Lipschitzian with the Lipschitz constant $\max\{|M_1|, |M_2|, \ldots, |M_k|\}$ and that its derivative $f'(x) = M_i$ for $x \in \Omega_i$ for $i = 1, 2, \ldots, k$. 
Recall that the **rank** of a matrix $A \in \mathbb{R}^{m \times n}$ is the maximum number of the rows of $A$ that are linearly independent.

Now, consider two matrices $P, Q \in \mathbb{R}^{m \times n}$ with $\text{rank}(P - Q) = 1$.

Observe then that the set $H_{PQ} = \{ x \in \mathbb{R}^n : Px = Qx \}$ is a hyperplane.

We call it the **hyperplane of the continuous contact** of the matrices $P$ and $Q$. 
The Ray-Fish Construction: An Exercise

Consider two matrices \( P, Q \in \mathbb{R}^{m \times n} \) with \( \text{rank}(P - Q) = 1 \).

Then there exists a row vector \( u^T \in \mathbb{R}^{1 \times n} \) such that

\[
H_{PQ} = \{ x \in \mathbb{R}^n : u^T x = 0 \}
\]

Define the mapping \( f : \mathbb{R}^n \to \mathbb{R}^m \) by

\[
f(x) = \begin{cases} 
    Px, & \text{if } u^T x \leq 0 \\
    Qx, & \text{if } u^T x \geq 0
\end{cases}
\]

and observe that \( f \) is Lipschitzian and piecewise linear.
The Ray-Fish Construction: Ray-Fish Lemma 1

Let $\alpha > 0$ and let $P, Q \in \mathbb{R}^{m \times n}$ be two matrices with $\text{rank}(P - Q) \leq 1$.

Then there exists a finitely piecewise affine Lipschitzian mapping $f_{\alpha, PQ}: \mathbb{R}^n \to \mathbb{R}^m$ such that

- $f_{\alpha, PQ}(x) = Px$ for all $x \in \mathbb{R}^n \setminus B_n$, where $B_n$ denotes the closed unit ball in $\mathbb{R}^n$,
- $f'_{\alpha, PQ}(x) = Q$ for all $x \in \Omega_{\alpha, PQ}$, where $\Omega_{\alpha, PQ} \subset B_n$ is a non-empty open set,
- and

$$\text{dist}(f'_{\alpha, PQ}(x), \{P, Q\}) < \alpha$$

whenever $f_{\alpha, PQ}$ is differentiable at $x \in \mathbb{R}^n$. 
The Ray-Fish Construction

Let \( \alpha > 0 \) and let \( P, Q \in \mathbb{R}^{m \times n} \) be two matrices with \( \text{rank}(P - Q) \leq 1 \).

- If \( \text{rank}(P - Q) = 0 \), then \( P = Q \) and the mapping \( f_{\alpha,PQ}(x) := Px \) works.

- Consider that \( \text{rank}(P - Q) = 1 \) in the following therefore.

Consider the contact hyperplane \( H_{PQ} = \{ x \in \mathbb{R}^n : Px = Qx \} \).

Let \( S^{n-1} \) denote the unit sphere in \( \mathbb{R}^n \).

Pick a vector \( u \in S^{n-1} \) such that \( u \perp H_{PQ} \).

Notice that \( H_{PQ} = \{ x \in \mathbb{R}^n : u^T x = 0 \} \).
The Ray-Fish Construction

Let \( \alpha > 0 \) and let \( P, Q \in \mathbb{R}^{m \times n} \) be two matrices with \( \text{rank}(P - Q) = 1 \).

Consider any \( \delta \in (0, 1) \) and define \( H_{\delta, PQ} := \{ x \in \mathbb{R}^n : u^T x = -\delta \} \).

Choose any points \( r_{0,1}, r_{0,2}, \ldots, r_{0,n} \in H_{PQ} \cap \mathbb{S}^{n-1} \) such that their convex hull \( \text{co}\{r_{0,1}, r_{0,2}, \ldots, r_{0,n}\} \) is a regular \((n - 1)\)-dimensional simplex.

Let \( r_{\delta,j} \in H_{\delta,PQ} \cap \mathbb{S}^{n-1} \) be the unique point such that \( \lambda r_{\delta,j} + (1 - \lambda)u = \mu r_{0,j} \) for some \( \lambda, \mu \in (0, 1) \) for \( j = 1, 2, \ldots, n \).

If \( n = 1 \), then let \( u := 1, r_{0,1} := 0 \) and \( r_{\delta,q} := -\delta \).
Let $\alpha > 0$ and let $P, Q \in \mathbb{R}^{m \times n}$ be two matrices with $\text{rank}(P - Q) = 1$.

Define

$$f_{\alpha,PQ}(x) = \begin{cases} Px, & \text{if } x \in \mathbb{R}^n \setminus \text{co}\{r_{\delta,1}, ..., r_{\delta,n}, u\} \supset \mathbb{R}^n \setminus \mathbb{B}_n \\ Qx - \delta(P - Q)u, & \text{if } x \in \text{co}\{r_{\delta,1}, ..., r_{\delta,n}, 0\} \\ M_{\delta,j}x - \delta(P - Q)u, & \text{if } x \in \text{co}\{r_{\delta,1}, ..., r_{\delta,j-1}, u, r_{\delta,j+1}, ..., r_{\delta,n}\} \end{cases}$$

where the matrices $M_{\delta,1}, M_{\delta,2}, ..., M_{\delta,n}$ are to be found so that the mapping $f_{\alpha,PQ}$ is well-defined, hence continuous and Lipschitzian.
Let $\alpha > 0$ and let $P, Q \in \mathbb{R}^{m \times n}$ be two matrices with $\text{rank}(P - Q) = 1$.

A few elementary calculations show that

$$M_{\delta,j} \rightarrow P \quad \text{as} \quad \delta \downarrow 0$$

Find $\delta \in (0, 1)$ so small that

$$\|M_{\delta,j} - P\| < \alpha \quad \text{for} \quad j = 1, \ldots, n$$

We are done thus.

The Ray-Fish Construction

Let $\alpha > 0$ and let $P, Q \in \mathbb{R}^{m \times n}$ be two matrices with $\text{rank}(P - Q) = 1$.

A few elementary calculations show that

$$M_{\delta,j} \rightarrow P \quad \text{as} \quad \delta \downarrow 0$$

Find $\delta \in (0, 1)$ so small that

$$\|M_{\delta,j} - P\| < \alpha \quad \text{for} \quad j = 1, \ldots, n$$

We are done thus.
Ray-Fish

Source: https://commons.wikimedia.org/wiki/File:Rays_(32199123686).jpg
Ray-Fish

Source: https://commons.wikimedia.org/wiki/File:Rays_(32088560952).jpg
The Ray-Fish

\[ u \]

\[ \delta \]

\[ r_{\delta,1} \quad r_{\delta,2} \]

\[ P_x \]

\[ M_{\delta,1}x - \delta(P - Q)u \]

\[ M_{\delta,2}x - \delta(P - Q)u \]

\[ Qx - \delta(P - Q)u \]
Let $\alpha > 0$ and let $P, Q \in \mathbb{R}^{m \times n}$ be two matrices with $\text{rank}(P - Q) = 1$.

Notice that:

- $f_{\alpha,PQ}'(x) \in \{Q, P, M_{\delta,1}, M_{\delta,2}\}$ whenever $f_{\alpha,PQ}$ is differentiable at $x \in \mathbb{R}^n$
- $\partial f_{\alpha,PQ}(0) = \text{co}\{Q, M_{\delta,1}, M_{\delta,2}\}$
- for any subspace $\{0\} \subset W \subset \mathbb{R}^n$,
  \[
  \partial(f_{\alpha,PQ})_W(0) = \text{co}\{Q_W, (M_{\delta,1})_W, (M_{\delta,2})_W\}\]
The Ray-Fish

\[ u \]

\[ \delta \]

\[ r_{\delta,1} \]

\[ r_{\delta,2} \]

\[ P_x \]

\[ M_{\delta,1}x - \delta(P - Q)u \]

\[ M_{\delta,2}x - \delta(P - Q)u \]

\[ Qx - \delta(P - Q)u \]

\[ \Omega_{\alpha,P,Q} \]
Let $\beta > 0$ and let $Q_0, \ldots, Q_k \in \mathbb{R}^{m \times n}$ be with $\text{rank}(Q_j - Q_{j+1}) \leq 1$ for $j = 0, \ldots, k - 1$.

Then there exists a Lipschitzian mapping $g_{\beta, Q_0 \ldots Q_k} : \mathbb{R}^n \to \mathbb{R}^m$ such that it is finitely piecewise affine and

- $g_{\beta, Q_0 \ldots Q_k}(x) = Q_0 x$ for all $x \in \mathbb{R}^n \setminus \mathbb{B}_n$,
- $g'_{\beta, Q_0 \ldots Q_k}(x) = Q_k$ for all $x$ from a non-empty open set $\Omega_{\beta, Q_0 \ldots Q_k} \subset \mathbb{B}_n$,
- and
  $$\text{dist}(g'_{\beta, Q_0 \ldots Q_k}(x), \{Q_0, \ldots, Q_k\}) < \beta$$

whenever $g_{\beta, Q_0 \ldots Q_k}$ is differentiable at $x \in \mathbb{R}^n$. 
The Recursive Ray-Fish Construction

\[ P := Q_0 \]
\[ Q := Q_1 \]
The Recursive Ray-Fish Construction

\[ P := Q_1 \]
\[ Q := Q_2 \]
The Recursive Ray-Fish Construction

\[ P := Q_2 \]
\[ Q := Q_3 \]
The Recursive Ray-Fish Construction

... and so on ...
The Recursive Ray-Fish Construction

\[ P := Q_{k-1} \]
\[ Q := Q_k \]
The Recursive Ray-Fish Construction:

The Result:
The Recursive Ray-Fish Construction:  Ray-Fish Colony Lemma 2

Let $\beta > 0$ and let $Q_0, \ldots, Q_k \in \mathbb{R}^{m \times n}$ be with $\text{rank}(Q_j - Q_{j+1}) \leq 1$ for $j = 0, \ldots, k - 1$.

Recall that, as we immerse into the (shifted and scaled) open sets

$\Omega_{\alpha,Q_0 Q_1}, \Omega_{\alpha,Q_1 Q_2}, \ldots, \Omega_{\alpha,Q_{k-1} Q_k}$, we encounter all the derivatives $Q_0, Q_1, \ldots, Q_k$.

Recall also that the matrices "$M_\delta,\ldots$" tend to the matrices $Q_0, Q_1, \ldots, Q_k$ as $\delta \downarrow 0$. 
The Recursive Ray-Fish Construction: Ray-Fish Colony Lemma 2

Let $\beta > 0$ and let $Q_0, \ldots, Q_k \in \mathbb{R}^{m \times n}$ be with $\text{rank}(Q_j - Q_{j+1}) \leq 1$ for $j = 0, \ldots, k - 1$.

Since the matrices "$M_{\delta, \ldots}$" tend to the matrices $Q_0, Q_1, \ldots, Q_k$ as $\delta \downarrow 0$,
it follows that

$$\text{dist}(g'_{\beta, Q_0 \ldots Q_k}(x), \{Q_0, \ldots, Q_k\}) < \beta$$

whenever $g_{\beta, Q_0 \ldots Q_k}$ is differentiable at $x \in \mathbb{R}^n$.

Notice that here we have

$$\Omega_{\beta, Q_0 \ldots Q_k} := \text{the (very small, shifted and scaled) } \Omega_{\alpha, Q_{k-1} Q_k}$$
The Ray-Fish Colony for a Line Segment

Until now, we have considered matrices $P, Q \in \mathbb{R}^{m \times n}$ or $Q_0, Q_1, ..., Q_k \in \mathbb{R}^{m \times n}$ with
\[
\text{rank}(P - Q) \leq 1 \quad \text{or} \quad \text{rank}(Q_j - Q_{j+1}) \leq 1 \quad \text{for} \quad j = 0, 1, ..., k - 1.
\]

It is now our purpose to construct the “ray-fish colony” mapping of analogous properties for general matrices $A, B \in \mathbb{R}^{m \times n}$. 
The Ray-Fish Colony for a Line Segment

Consider two matrices \( U, V \in \mathbb{R}^{m \times n} \) consisting of rows \( u_1, \ldots, u_m \in \mathbb{R}^{1 \times n} \) and \( v_1, \ldots, v_m \in \mathbb{R}^{1 \times n} \), respectively.

Let \( T_{UV}^i \) denote the \( m \times n \) matrix consisting of the first \( (m - i) \) rows \( u_1, \ldots, u_{m-i} \) of matrix \( U \) and of the last \( i \) rows \( v_{m-i+1}, \ldots, v_m \) of matrix \( V \) for \( i = 0, 1, \ldots, m \):

\[
T_{UV}^i = \begin{pmatrix}
\vdots & u_1 & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & u_{m-i} & \vdots \\
\vdots & v_{m-i+1} & \vdots \\
\vdots & v_{m-i+2} & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & v_m & \vdots 
\end{pmatrix}
\]

Observe that

\[
\text{rank}(T_{UV}^i - T_{UV}^{i+1}) \leq 1 \quad \text{for} \quad i = 0, \ldots, m - 1
\]
The Ray-Fish Colony for a Line Segment

Consider two matrices $U, V \in \mathbb{R}^{m \times n}$:

$$
U = \begin{pmatrix}
\vdots & u_1 & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & u_{m-i} & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & u_m & \vdots 
\end{pmatrix} \quad T_{UV}^i = \begin{pmatrix}
\vdots & u_1 & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & u_{m-i} & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & v_m & \vdots 
\end{pmatrix} \quad V = \begin{pmatrix}
\vdots & v_1 & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & v_{m-i} & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & v_m & \vdots 
\end{pmatrix}
$$

Observe also that

$$
T_{UV}^0 = U \quad \text{and} \quad T_{UV}^m = V
$$

and

$$
\max\{\|T_{UV}^i - U\|, \|T_{UV}^i - V\|\} \leq \|U - V\| \quad \text{for} \quad i = 0, 1, \ldots, m
$$
The Ray-Fish Colony for a Line Segment

Consider any matrices $A, B \in \mathbb{R}^{m \times n}$.

Recall that the line segment between the matrices $A$ and $B$ is the convex hull

$$[A, B] := \text{co}\{A, B\}$$

Considering a positive natural number $\ell$,

divide the line segment by $(\ell - 1)$ points $S_1, ..., S_{\ell - 1}$ equidistantly:

$$A = S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow \cdots \rightarrow S_{\ell - 1} \rightarrow S_\ell = B$$
The Ray-Fish Colony for a Line Segment

Notice that

\[
\text{dist} \left( T_{S_jS_{j+1}}^i, [A, B] \right) \leq \max \left\{ \| T_{S_jS_{j+1}}^i - S_j \|, \| T_{S_jS_{j+1}}^i - S_{j+1} \| \right\} \leq \\
\leq \| S_j - S_{j+1} \| = \frac{1}{\ell} \| A - B \| \downarrow 0 \quad \text{as} \quad \ell \to \infty
\]
The Ray-Fish Colony for a Line Segment

Rename this cortege of matrices

\[
\begin{array}{cccccc}
T_{S_0S_1}^1 & \ldots & T_{S_0S_1}^{m-1} & T_{S_1S_2}^1 & \ldots & T_{S_1S_2}^{m-1} & T_{S_2S_3}^1 & \ldots & T_{S_{\ell-2}S_{\ell-1}}^{m-1} & T_{S_{\ell-1}S_{\ell}}^1 & T_{S_{\ell-1}S_{\ell}}^{m-1}
\end{array}
\]

\[A = S_0 \quad S_1 \quad S_2 \quad \ldots \quad S_{\ell-1} \quad S_\ell = B\]

to

\[
Q_0 \quad Q_1 \quad \ldots \quad Q_{m-1} \quad Q_m \quad \ldots \quad Q_{2m} \quad \ldots \quad \ldots \quad Q_{(\ell-1)m} \quad \ldots \quad Q_{\ell m}
\]

Recall that

\[
dist(Q_j, [A, B]) \leq \frac{1}{\ell} \|A - B\| \downarrow 0 \quad \text{as} \quad \ell \to \infty
\]

and notice that

\[
\text{rank}(Q_j - Q_{j+1}) \leq 1 \quad \text{for} \quad j = 0, \ldots, \ell m - 1
\]

By applying the Ray-Fish Colony Construction, we obtain:
Ray-Fish Colony for a Line Segment: Lemma 3

Let $\gamma > 0$ and let $A, B \in \mathbb{R}^{m \times n}$ be any matrices.

Then there exists a finitely piecewise affine Lipschitzian mapping $h_{\gamma,[A,B]} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

- $h_{\gamma,[A,B]}(x) = Ax$ for all $x \in \mathbb{R}^n \setminus B_n$,
- $h'_{\gamma,[A,B]}(x) = B$ for all $x$ from a non-empty open set $\Omega_{\gamma,[A,B]} \subset B_n$,
- and
  \[
  \text{dist}(h'_{\gamma,[A,B]}(x), [A, B]) < \gamma
  \]
  whenever $h_{\gamma,[A,B]}$ is differentiable at $x \in \mathbb{R}^n$. 
The Ray-Fish Colony for a Polygonal Chain

Consider any matrices $B_0, B_1, \ldots, B_N \in \mathbb{R}^{m \times n}$.

Recall that the polygonal chain $[B_0, B_1, \ldots, B_N]$ is a curve which consists of the line segments connecting the consecutive vertices, that is the union of the convex hulls

$$[B_0, B_1, \ldots, B_N] := \text{co}\{B_0, B_1\} \cup \text{co}\{B_1, B_2\} \cup \cdots \cup \text{co}\{B_{N-1}, B_N\}$$
Given the matrices $B_0, B_1, ..., B_N \in \mathbb{R}^{m \times n}$, consider a positive natural $\ell$,

- divide each of the line segments $[B_0, B_1], [B_1, B_2], ..., [B_{N-1}, B_N]$ by $(\ell - 1)$ points $S_1^1, ..., S_{\ell-1}^1, S_1^2, ..., S_{\ell-1}^2, ..., S_1^N, ..., S_{\ell-1}^N$, equidistantly,
- consider also the intervening transitional matrices:

$$B_0 = S_0^1 = T_0^0 S_0^1 S_1^1, ..., T_m^1 S_0^1 S_1^1 = S_1^1 = T_0^1 S_1^1 S_2^1, ..., T_m^1 S_1^1 S_2^1 = S_1^2 = T_0^1 S_2^1 S_3^1, ..., T_m^1 S_{\ell-1}^1 S_\ell^1 = S_\ell^1 = B_1$$

$$B_1 = S_0^2 = T_0^0 S_0^2 S_1^2, ..., T_m^2 S_0^2 S_1^2 = S_1^2 = T_0^2 S_1^2 S_2^2, ..., T_m^2 S_1^2 S_2^2 = S_2^2 = T_0^2 S_2^2 S_3^2, ..., T_m^2 S_{\ell-1}^2 S_\ell^2 = S_\ell^2 = B_2$$

$$...$$

$$B_{N-1} = S_0^N = T_0^0 S_0^N S_1^N, ..., T_m^N S_0^N S_1^N = S_1^N = T_0^N S_1^N S_2^N, ..., T_m^N S_1^N S_2^N = S_2^N = T_0^N S_2^N S_3^N, ..., T_m^N S_{\ell-1}^N S_\ell^N = S_\ell^N = B_N$$

- and rename this long cortege of matrices to $Q_0, ..., Q_{\ell m N}$. 

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The Ray-Fish Colony for a Polygonal Chain
The Ray-Fish Colony for a Polygonal Chain

We then have:
\[ \text{dist}(Q_j, [B_0, B_1, \ldots, B_N]) \leq \frac{1}{\ell} \max\{\|B_0 - B_1\|, \ldots, \|B_{N-1} - B_N\|\} \downarrow 0 \quad \text{as} \quad \ell \to \infty \]

Notice also that
\[ \text{rank}(Q_j - Q_{j+1}) \leq 1 \quad \text{for} \quad j = 0, \ldots, \ell mN - 1 \]

By applying the Ray-Fish Colony Construction, we obtain:
The Ray-Fish Colony for a Polygonal Chain: Lemma 4

Let $\gamma > 0$ and let $B_0, B_1, ..., B_N \in \mathbb{R}^{m \times n}$ be any matrices.

Then there exists a Lipschitzian mapping $h_{\gamma, [B_0, B_1, ..., B_N]} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that it is finitely piecewise affine and

- $h_{\gamma, [B_0, B_1, ..., B_N]}(x) = B_0 x$ for all $x \in \mathbb{R}^n \setminus \mathbb{B}_n$,
- $h_{\gamma, [B_0, B_1, ..., B_N]}'(x) = B_N$ for all $x$ from a non-empty open set $\Omega_{\gamma, [B_0, B_1, ..., B_N]} \subset \mathbb{B}_n$,
- and

$$\text{dist}(h_{\gamma, [B_0, B_1, ..., B_N]}(x), [B_0, B_1, ..., B_N]) < \gamma$$

whenever $h_{\gamma, [B_0, B_1, ..., B_N]}$ is differentiable at $x \in \mathbb{R}^n$. 
Ray-Fish Colony Construction for a Polygonal Chain

Notice that the open set
\[ \Omega_{\gamma,[B_0,B_1,\ldots,B_N]} \subset \mathbb{B}_n \] is the very last open set inside the very last ray-fish inside.

[Diagram showing a polygonal chain with rays and open sets labeled accordingly]
The Corona Construction

For a $\gamma > 0$ and for $B_0, B_1, \ldots, B_N \in \mathbb{R}^{m \times n}$, we have constructed (recursively) the Ray-Fish Colony $h_{\gamma,[B_0,B_1,\ldots,B_N]}: \mathbb{R}^n \to \mathbb{R}^m$ for the polygonal chain $[B_0, B_1, \ldots, B_N]$.

The non-empty open set $\Omega_{\gamma,[B_0,B_1,\ldots,B_N]} \subset \mathbb{B}_n$ is very small:

Now, make many copies of this ray-fish colony $h_{\gamma,[B_0,B_1,\ldots,B_N]}$, shift them and place them beyond the sphere of radius 3, say, in such a way that each ray emanating from the origin passes through at least one of the (shifted) open sets $\Omega_{\gamma,[B_0,B_1,\ldots,B_N]}$, the colonies being pairwise disjoint.
The Corona Construction
The Corona Construction

Plenty (finitely many) of copies of the ray-fish colony \( \bullet = h_{\gamma,[B_0,B_1,...,B_N]} \) are placed into a spherical shell \( T(\rho, P) = \{ x \in \mathbb{R}^n : \rho \leq ||x|| \leq P \} \) in such a way that:

- the colonies are pairwise disjoint, and
- each ray emanating from the origin passes through at least one of the (shifted) tiny open set \( \bullet = \Omega_{\gamma,[B_0,B_1,...,B_N]} \) for \( 0 < \rho < P < +\infty \).

We thus obtain:
Ray-Fish Corona Construction for a Polygonal Chain: Lemma 5

Let $\delta > 0$ and let $B_0, B_1, ..., B_N \in \mathbb{R}^{m \times n}$ be any matrices. Then there exist numbers $0 < \rho_{\delta, [B_0, B_1, ..., B_N]} < P_{\delta, [B_0, B_1, ..., B_N]} < +\infty$ and a finitely piecewise affine Lipschitzian mapping $\varphi_{\delta, [B_0, B_1, ..., B_N]} : \mathbb{R}^n \to \mathbb{R}^m$ such that

- $\varphi_{\delta, [B_0, B_1, ..., B_N]}(x) = B_0 x$ for all $x \in \mathbb{R}^n \setminus T(\rho_{\delta, [B_0, B_1, ..., B_N]}, P_{\delta, [B_0, B_1, ..., B_N]})$,
- for every subspace $\{0\} \not\subset W \subset \mathbb{R}^n$, there are a $w \in W$ and a $\lambda > 0$ such that the ball $B(w, \lambda) \subset T(\rho_{\delta, [B_0, B_1, ..., B_N]}, P_{\delta, [B_0, B_1, ..., B_N]})$ and
  $\varphi_{\delta, [B_0, B_1, ..., B_N]}'(x) = B_N$ for all $x \in W \cap B(w, \lambda)$,
- and
  \[ \text{dist}(\varphi_{\delta, [B_0, B_1, ..., B_N]}(x), [B_0, B_1, ..., B_N]) < \delta \]
  whenever $\varphi_{\delta, [B_0, B_1, ..., B_N]}$ is differentiable at $x \in \mathbb{R}^n$. 
Finale
Given a non-empty compact convex set \( \mathcal{P} \subset \mathbb{R}^{m \times n} \) of matrices, it is our purpose to construct a Lipschitzian mapping \( g: \mathbb{R}^n \to \mathbb{R}^m \) such that

\[
\partial g(0) = \mathcal{P}
\]

actually

\[
\partial g|_W(0) = \mathcal{P}|_W \quad \text{for every linear subspace} \quad \{0\} \not\subset W \subset \mathbb{R}^n
\]
Finale

The non-empty compact convex set $\mathcal{P} \subset \mathbb{R}^{m \times n}$ of matrices is separable. Therefore, there exists a countable sequence

$$B_0, B_1, B_2, B_3, B_4, B_5, \ldots \in \mathcal{P}$$

such that the convex hull

$$\text{co}\{B_0, B_1, B_2, B_3, B_4, B_5, \ldots\}$$

of the set is dense in $\mathcal{P}$.

(Remark: If already $\text{co}\{B_0, B_1, \ldots, B_N\}$ is dense in $\mathcal{P}$, then consider $B_0, B_1, \ldots, B_N, B_0, B_1, \ldots, B_N, B_0, B_1, \ldots, B_N, \ldots$)
Given the non-empty compact convex set \( \mathcal{P} \subset \mathbb{R}^{m \times n} \) of matrices and having the countably infinite sequence \( B_0, B_1, B_2, B_3, B_4, B_5, \ldots \in \mathcal{P} \), consider the longer and longer polygonal chains

\[
[B_0, B_1] \\
[B_0, B_1, B_2] \\
[B_0, B_1, B_2, B_3] \\
[B_0, B_1, B_2, B_3, B_4] \\
[B_0, B_1, B_2, B_3, B_4, B_5] \\
\text{and so on}
\]
Given the non-empty compact convex set $\mathcal{P} \subset \mathbb{R}^{m \times n}$ of matrices and having polygonal chains $[B_0, B_1, \ldots, B_N]$ for each $N \in \mathbb{N}$, consider also a decreasing sequence

$$\delta_1 > \delta_2 > \delta_3 > \delta_4 > \delta_5 > \ldots > 0$$

such that

$$\delta_N \downarrow 0 \quad \text{as} \quad N \to \infty$$
Given the non-empty compact convex set \( \mathcal{P} \subset \mathbb{R}^{m \times n} \) of matrices, having polygonal chains \([B_0, B_1, ..., B_N]\) for each \( N \in \mathbb{N} \), and having also the decreasing sequence \( \delta_N \downarrow 0 \), with \( N \in \mathbb{N} \), construct the Coronas for these polygonal chains with these "delta’s":

\[
\phi_{\delta_1,[B_0,B_1]}
\]
\[
\phi_{\delta_2,[B_0,B_1,B_2]}
\]
\[
\phi_{\delta_3,[B_0,B_1,B_2,B_3]}
\]
\[
\phi_{\delta_4,[B_0,B_1,B_2,B_3,B_4]}
\]
\[
\phi_{\delta_5,[B_0,B_1,B_2,B_3,B_4,B_5]}
\]
and so on
Finale

Given the non-empty compact convex set \( \mathcal{P} \subset \mathbb{R}^{m \times n} \) of matrices,

- take the first corona \( \varphi_{\delta_1,[B_0,B_1]} \),
- take the second corona \( \varphi_{\delta_2,[B_0,B_1,B_2]} \) and shrink it to be inside the first one
- take the third corona \( \varphi_{\delta_3,[B_0,B_1,B_2,B_3]} \) and shrink it to be inside the second one
- take the fourth corona \( \varphi_{\delta_4,[B_0,B_1,B_2,B_3,B_4]} \) and shrink it to be inside the third one
- and so on

That is, the coronas tend to the origin.
We have thus obtained:
Let $m, n \in \mathbb{N}$ and let $\mathcal{P} \subset \mathbb{R}^{m \times n}$ be any non-empty compact convex set of matrices.

Then there exists a countably piecewise affine Lipschitzian mapping $g: \mathbb{R}^n \to \mathbb{R}^m$, with $g(0) = 0$, such that, for every linear subspace $\{0\} \subsetneq W \subset \mathbb{R}^n$, the Clarke Jacobian

$$\partial g_W(0) = \mathcal{P}_W$$

Thank You for your attention