Openness, Hölder metric regularity and Hölder continuity properties of semialgebraic set-valued maps

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Outline

Introduction

In Variational Analysis, the following notions are well-known:

- Openness
- Metric regularity
- Lipschitz/Hölder continuity

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See the books and the references therein:

Aubin–Frankowska (1990), Bonnas–Shapiro (2000), Ioffe (2017), Klatte–Kummer (2002), Mordukhovich (2006), Rockafellar–Wets (1998)

Theorem (Penot, Nonlinear Anal., 1989)

Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued map. The following are equivalent:

(i) *the map F is open at a linear rate;*

(ii) *the map F is metrically regular;*

(iii) *the inverse map F^{-1} is pseudo-Lipschitz continuous (i.e., F^{-1} has the Aubin property).*

Theorem (Borwein–Zhuang, J. Math. Anal. Appl., 1988)

Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued map. The following are equivalent:

(i) *the map F is open at an order rate $p > 0$;*

(ii) *the map F is metrically regular of order $1/p$;*

(iii) *the inverse map F^{-1} is pseudo-Hölder of order $1/p$.*

Introduction

Note: Openness at a positive-order rate implies openness;

Note: Openness at a positive-order rate implies openness; the converse is not true in general:

Example

Let

$$f : \mathbb{R} \to (-1, 1), \quad x \mapsto f(x) := \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -e^{1/x} & \text{if } x < 0. \end{cases}$$

We have f is open but f is not open at a positive-order rate.

Introduction

Theorem (Gowda–Sznajder, Math. Program., 1996)

Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a polyhedral set-valued map with its range being a convex set. The following are equivalent:

(i) *the map F is open;*

(ii) *the inverse map F^{-1} is lower Lipschitz continuous;*

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Polyhedral maps are semialgebraic!

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Polyhedral maps are semialgebraic! So it is natural to study relations between the following notions for **semialgebraic maps**:

- openness,
- Hölder metric regularity, and
- Hölder continuity properties

Definitions

- A set $X \subset \mathbb{R}^n$ is **locally closed** if for each $x \in X$, there exists $\epsilon > 0$ such that $\mathbb{B}_\epsilon(x) \cap X$ is a closed set in $\mathbb{R}^n$.

- Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued map. Set

$$\begin{aligned} \textbf{dom}F &:= \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}, \\ \textbf{range}F &:= \{y \in F(x) \mid x \in \mathrm{dom}F\}, \\ \textbf{graph}F &:= \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid x \in \mathrm{dom}F, y \in F(x)\}. \end{aligned}$$

- The **inverse map** $F^{-1}\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ of the map F is defined as

$$F^{-1}(y) := \{x \in \mathbb{R}^m \mid y \in F(x)\}.$$

Definitions

• The map F is called **lower semicontinuous** (l.s.c) if for any $x \in \mathrm{dom} F$, any $y \in F(x)$ and for any sequence $\{x^k\} \subset \mathrm{dom} F$ converging to x, there exists a sequence $\{y^k\} \subset F(x^k)$ converging to y.

• F is said to be an **open map** from $\mathrm{dom} F$ into $\mathrm{range} F$ if for every open set U in $\mathrm{dom} F$, the set $F(U)$ is open in $\mathrm{range} F$.

Definitions

- F is said to be **Hölder metrically regular** if for each point $y^* \in \mathrm{range} F$ and for each compact set $K \subset \mathbb{R}^n$, there exist constants $\epsilon > 0$, $c > 0$ and $\alpha > 0$ such that

$$\mathrm{dist}(x, F^{-1}(y)) \leq c[\mathrm{dist}(y, F(x))]^{\alpha}$$

for all $x \in K$ and all $y \in \mathbb{B}_{\epsilon}(y^*) \cap \mathrm{range} F$.

- F is said to be **Hölder metrically subregular** if for each point $y^* \in \mathrm{range} F$ and for each compact set $K \subset \mathbb{R}^n$, there exist constants $c > 0$ and $\alpha > 0$ such that

$$\mathrm{dist}(x, F^{-1}(y^*)) \leq c[\mathrm{dist}(y^*, F(x))]^{\alpha}$$

for all $x \in K$.

Definitions

- F is said to be **pseudo-Hölder continuous** if for each point $x^* \in \mathrm{dom} F$ and for each compact set $K \subset \mathbb{R}^m$, there exist constants $\epsilon > 0$, $c > 0$ and $\alpha > 0$ such that

$$F(x^1) \cap K \quad \subset \quad F(x^2) + c\|x^1 - x^2\|^\alpha \mathbb{B}$$

for all $x^1, x^2 \in \mathbb{B}_\epsilon(x^*) \cap \mathrm{dom} F$.

- F is said to be **lower pseudo-Hölder continuous** if for each point $x^* \in \mathrm{dom} F$ and for each compact set $K \subset \mathbb{R}^m$, there exist constants $\epsilon > 0$, $c > 0$ and $\alpha > 0$ such that

$$F(x^*) \cap K \quad \subset \quad F(x) + c\|x^* - x\|^\alpha \mathbb{B}$$

for all $x \in \mathbb{B}_\epsilon(x^*) \cap \mathrm{dom} F$.

Definitions

• A subset S of $\mathbb{R}^n$ is called **semialgebraic**, if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid f_i(x) = 0,\ i = 1, \ldots, k; f_i(x) > 0,\ i = k+1, \ldots, p\},$$

where all f_i are polynomials.

• A set-valued map $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is said to be **semialgebraic**, if its graph is a semialgebraic set.

Examples

The following maps are semialgebraic:

- Piecewise linear/quadratic maps
- Polynomial/rational maps

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Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a semialgebraic map. Then

- The inverse map $F^{-1}\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is semialgebraic
- The following maps are semialgebraic

$$\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \quad (x, y) \mapsto \mathrm{dist}(x, F^{-1}(y)),$$

and

$$\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \quad (x, y) \mapsto \mathrm{dist}(y, F(x)).$$

Proposition (The Łojasiewicz inequality)

Let $K \subset \mathbb{R}^n$ be a compact semialgebraic set and let $\phi, \psi \colon K \to \mathbb{R}$ be semialgebraic functions satisfying the following:

(i) *ϕ is continuous;*

(ii) *for any sequence $\{x^k\} \subset K$ converging to $\bar{x} \in K$ such that $\lim_{k\to\infty} \psi(x^k) = 0$, it holds that $\psi(\bar{x}) = 0$.*

Then $\psi^{-1}(0) \subset \phi^{-1}(0)$ if and only if there exist constants $c > 0$ and $\alpha > 0$ such that

$$c|\psi(x)|^{\alpha} \geq |\phi(x)| \quad \textit{for all} \quad x \in K.$$

Note: If ψ is continuous then (ii) holds.

Theorem

Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a semialgebraic set-valued map with closed graph. The map F is Hölder metrically subregular.

Tool: The Łojasiewicz inequality!

Characterizations of the openness

Proof.

Fix a point $y^* \in \mathrm{range} F$ and a compact semialgebraic set $K \subset \mathbb{R}^n$. Consider the functions

$$\phi \colon K \to \mathbb{R}, \quad x \mapsto \mathrm{dist}(x, F^{-1}(y^*)),$$

and

$$\psi \colon K \to \mathbb{R}, \quad x \mapsto \mathrm{dist}(y^*, F(x)).$$

Applying the Łojasiewicz inequality to the functions ϕ and ψ, we can see that the map F is Hölder metrically subregular. □

Theorem

Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a semialgebraic set-valued map with closed graph. The following are equivalent:

(i) *F is an open map from $\mathrm{dom} F$ into $\mathrm{range} F$ and $\mathrm{range} F$ is locally closed;*

(ii) *F is Hölder metrically regular;*

(iii) *F^{-1} is pseudo-Hölder continuous;*

(iv) *F^{-1} is lower pseudo-Hölder continuous.*

The main point of the proof is to show the implication (i) $\Rightarrow$ (ii).

Tool: The Łojasiewicz inequality!

Proof of (i) ⇒ (ii).

There are two steps:

1. F is open if and only if the following function is continuous:

$$\phi \colon \mathbb{R}^n \times \mathrm{range} F \to \mathbb{R}, \quad (x, y) \mapsto \mathrm{dist}(x, F^{-1}(y)).$$

Proof of (i) $\Rightarrow$ (ii).

There are two steps:

1. F is open if and only if the following function is continuous:

$$\phi\colon \mathbb{R}^n \times \mathrm{range}F \to \mathbb{R}, \quad (x, y) \mapsto \mathrm{dist}(x, F^{-1}(y)).$$

2. Applying the Łojasiewicz inequality to the functions

$$\phi(x, y) := \mathrm{dist}(x, F^{-1}(y)) \quad \text{and} \quad \psi(x, y) := \mathrm{dist}(y, F(x)),$$

we can see that the map F is Hölder metrically regular. □

Corollary

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous semialgebraic (single valued) map. Then the following are equivalent:

(i) *f is an open map from $\mathbb{R}^n$ into $\mathrm{range} f$ and $\mathrm{range} f$ is locally closed;*

(ii) *f is Hölder metrically regular;*

(iii) *f^{-1} is pseudo-Hölder continuous;*

(iv) *f^{-1} is lower pseudo-Hölder continuous.*

Proof.

Since the map f is continuous, the graph of f is closed. Then the conclusion follows directly from the previous theorem. □

Proposition (See also Pühl, J. Math. Anal. Appl., 1998)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a **(not necessarily semialgebraic)** *continuous function. Then the following are equivalent:*

(i) *f is open;*

(ii) *f has no extremum points.*

Proof.

This is based on the following facts:

1. If $X \subset \mathbb{R}^n$ is a compact and connected set, then so is $f(X)$.

2. Every compact and connected set in $\mathbb{R}$ is a closed and bounded interval. □

Consider the semialgebraic variational inequality:

$$\textbf{Find } \mathbf{x} \in \mathbf{C} \textbf{ s.t. } \langle \mathbf{f}(\mathbf{x}) + \mathbf{p}, \mathbf{y} - \mathbf{x} \rangle \ \geq \ \mathbf{0} \quad \forall \mathbf{y} \in \mathbf{C}, \qquad \text{(VI)}$$

where

- $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous semialgebraic map,
- $C \subset \mathbb{R}^n$ is a closed convex semialgebraic set, and
- $p \in \mathbb{R}^n$ is a parameter vector.

Let $\mathcal{S}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be the **solution map** associated to (VI) and let $\mathcal{F}\colon \mathbb{R}^n \to \mathbb{R}^n$ be the **normal map** defined by

$$\mathcal{F}(u) \;:=\; f(\Pi_C(u)) + u - \Pi_C(u),$$

where Π_C is the Euclidean projection onto the set C. We have

$$\begin{aligned} \mathcal{S}(p) &= \Pi_C\big(\mathcal{F}^{-1}(-p)\big), \\ \mathcal{F}^{-1}(-p) &= \{x - f(x) - p \mid x \in \mathcal{S}(p)\}. \end{aligned}$$

For simplicity, we will assume that $\mathrm{dom}\mathcal{S}$ and $\mathrm{range}\mathcal{S}$ are open.

Semialgebraic variational inequalities: Affine case

Theorem (Dontchev–Rockafellar, SIOPT, 1996)

*If f **is an affine map and** C **is a convex polyhedral set**, then the following are equivalent:*

(i) $\mathcal{S}$ *is lower semicontinuous;*

(ii) $\mathcal{S}$ *is pseudo-Lipschitz continuous;*

(iii) $\mathcal{S}$ *is locally single valued and Lipschitz continuous;*

(iv) *The "critical face" condition holds.*

See also Ioffe [Math. Program, 2018] for a new proof!

Semialgebraic variational inequalities: General case

Dontchev–Rockafellar (SIOPT, 1996) ask whether **the lower semicontinuity of $\mathcal{S}$ implies the local single valuedness and Lipschitz continuity of $\mathcal{S}$.**

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Example

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (x^2 - y^2, 2xy)$ and $C := \mathbb{R}^2$. Then the solution map $\mathcal{S}$ associated to (VI) is given by

$$\mathcal{S}\colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2, \quad p \mapsto f^{-1}(-p).$$

The map $\mathcal{S}$ is l.s.c, but it is not locally single valued and also not Lipschitz continuous around $(0, 0) \in \mathbb{R}^2$.

Semialgebraic variational inequalities: General case

Theorem

The following statements are equivalent:

(i) $\mathcal{S}$ *is lower semicontinuous;*

(ii) $\mathcal{S}$ *is pseudo-Hölder continuous;*

(iii) $\mathcal{S}$ *is lower pseudo-Hölder continuous;*

(iv) $\mathcal{S}^{-1}$ *is open (i.e., it maps open sets into open sets);*

(v) $\mathcal{S}^{-1}$ *is Hölder metrically regular.*

(vi) $\mathcal{F}$ *is open;*

(vii) $\mathcal{F}$ *is Hölder metrically regular;*

(viii) $\mathcal{F}^{-1}$ *is pseudo-Hölder continuous;*

(ix) $\mathcal{F}^{-1}$ *is lower pseudo-Hölder continuous.*

Note: The lower semicontinuity of $\mathcal{S}$ does **not** imply the local single valuedness of $\mathcal{S}$.

Note: The lower semicontinuity of $\mathcal{S}$ does **not** imply the local single valuedness of $\mathcal{S}$. However, we have

Theorem

If the map $\mathcal{S}$ is l.s.c, then there is an integer N such that

$$\#\mathcal{S}(p) \ \leq \ N \quad \textit{for all} \quad p \in \mathbb{R}^n.$$

Proof.

The proof uses tools from Semialgebraic Geometry. □

References and thank you!

The details of the talk can be found in the manuscript:

- J. H. Lee and T. S. Phạm, Openness, Hölder metric regularity and Hölder continuity properties of semialgebraic set-valued maps, arxiv.org/abs/2004.02188, 2020.

Thank you very much for your kind attention!