Optimality conditions in convex (semi-)infinite optimization.

An approach based on the subdifferential of the supremum function.

Marco A. López

University of Alicante

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The problem

We deal with the *convex optimization problem*

$$(\mathcal{P}) \quad \text{Min} \quad g(x)$$
$$\text{s.t.} \quad f_t(x) \leq 0, \ t \in T,$$
$$x \in \mathbb{C},$$

where T is an arbitrary (possibly infinite) index set, $\mathbb{C}$ is a non-empty closed convex subset of a (separated) locally convex vector space X, and $g, f_t : X \to \mathbb{R} \cup \{+\infty\}$, $t \in T$, are proper lsc convex functions defined on X. We assume that the *constraint system*

$$\tau := \{f_t(x) \leq 0, \ t \in T; \ x \in \mathbb{C}\}, \tag{1}$$

is *consistent*; i.e., it has a non-empty *set of feasible solutions*, which is represented by $\mathbb{F}$.

An important particular case is that in which the explicit constraints are *affine and continuous* and there is *no constraint set* $\mathbb{C}$ (equivalently, $\mathbb{C} = X$), i.e.

$$\tau = \{\langle a_t^*, x\rangle \leq b_t,\ t \in T\}, \tag{2}$$

where $a_t^* \in X^*,\ t \in T$.

When T is *infinite* , the objective function g is *linear* , and $X = \mathbb{R}^n$, we are dealing with the so-called *linear semi-infinite optimization problem* (*LSIP*, in brief).

Our aims:

1. To study, in the framework of infinite convex systems, some *CQ's* as the *Farkas-Minkowski property* (FM, in brief) and the *local Farkas-Minkowski property* (LFM, in short).
2. To provide *optimality conditions* by appealing to the properties of the *supremum function* of an infinite family of convex functions and the characterizations of its subdifferential.
3. To formulate weak *CQ's* and derive associated *optimality conditions*.

Summary

- Associated unconstrained problems
- Notation and basic tools
- KKT'1 optimality conditions - FM property
- KKT'2 optimality conditions - LFM property
- KKT'3 asymptotic optimality conditions
- Subdifferential calculus for the sum and the supremum function
- KKT'4 conditions for SIP under compacity/(upper)continuity
- Bibliographical comments and references

Associated unconstrained problems

- Let $\bar{x} \in \mathbb{F}$, introduce the function

$$\varphi(x) := \sup\{g(x) - g(\bar{x});\ f_t(x),\ t \in T;\ \mathrm{I}_{\mathbb{C}}(x)\},$$

and consider the unconstrained problem

$$\inf \varphi,\ x \in X, .$$

- For $x \in \mathbb{F}$ one has $g(x) - g(\bar{x}) \geq 0$, and so $\varphi(x) \geq \varphi(\bar{x}) = 0$. Hence, $\bar{x}$ is optimal for $(\mathcal{P}) \implies \theta \in \partial\varphi(\bar{x})$.
- φ is a supremum function, and $\partial\varphi$ can be expressed using the *approximate/exact subdifferentials* of g and the f_t's.
- We also have that $\bar{x}$ is optimal for $(\mathcal{P})$ if and only if $\bar{x}$ is optimal for $\inf_{x \in X}(g + \mathrm{I}_{\mathbb{F}})(x)$.

Notations and basic tools

- X is a (real) Hausdorff locally convex space (Hlsc); X^* is its dual space; X and X^* are paired in duality by $\langle x, x^* \rangle$.
- Given $A, B \subset X$ (or in X^*), we consider the Minkowski sum:
  $A + B := \{a + b \mid a \in A, \ b \in B\}, \quad A + \emptyset := \emptyset + A := \emptyset.$
- $\operatorname{conv} A$ is the *convex hull* of A, $\operatorname{cone} A$ is the *convex cone generated by* A $(\operatorname{cone} \emptyset = \{\theta\})$, and $\operatorname{aff} A$ is the *affine hull* of A.
- $\operatorname{int} A$ is the *interior* of A, $\operatorname{cl} A$ and $\overline{A}$ denote indistinctly the *closure* of A (w^**–closure* if $A \subset X^*$); $\operatorname{rint} A$ is the topological *relative interior* of A (i.e., the interior of A in the topology relative to $\operatorname{aff} A$ if $\operatorname{aff} A$ is closed, and $\emptyset$ otherwise).
- $\mathrm{N}_A(x)$ is the *normal cone* to A at $x \in A$.
- A family of convex sets $\{A_i, \ i \in I\}$ such that $\cap_{i \in I} A_i \neq \emptyset$ has the *strong conical hull intersection property (the strong CHIP) at* $x \in \cap_{i \in I} A_i$ if

$$\mathrm{N}_{\cap_{i \in I} A_i}(x) = \sum\nolimits_{i \in I} \mathrm{N}_{A_i}(x)$$
$$:= \left\{ \sum\nolimits_{i \in J} a_i, \ a_i \in \mathrm{N}_{A_i}(x), \ J \text{ being finite subset of } I \right\}.$$

- Given $h : X \to \mathbb{R} \cup \{+\infty\}$, $\operatorname{dom} h$ and $\operatorname{epi} h$ represent its *(effective) domain* and *epigraph*, respectively.
- h is *proper* if $\operatorname{dom} h \neq \emptyset$; it is *convex* if $\operatorname{epi} h$ is convex, and $h \in \Gamma_0(X)$ if it is proper, lower semicontinuous and convex.
- $\overline{\operatorname{conv}} h$ represents the *lsc convex hull* of h; i.e., $\operatorname{epi}(\overline{\operatorname{conv}} h) = \overline{\operatorname{conv}}\,(\operatorname{epi} h)$.
- The *$\varepsilon-$subdifferential* of h at $x \in h^{-1}(\mathbb{R})$, $\varepsilon \geq 0$, is the w^*-closed convex set in X^*

$$\partial_\varepsilon h(x) := \{x^* \in X^* \mid h(y) - h(x) \geq \langle y - x, x^* \rangle - \varepsilon, \ \forall y \in X\}.$$

- If $\partial h(x) \neq \emptyset$,

$$h(x) = (\operatorname{cl} h)(x) \text{ and } \partial_\varepsilon h(x) = \partial_\varepsilon(\operatorname{cl} h)(x). \tag{3}$$

The *Legendre-Fenchel conjugate* of h is the lsc convex function $h^*: X^* \to \mathbb{R} \cup \{+\infty\}$ given by

$$h^*(x^*) := \sup\{\langle x, x^*\rangle - h(x) \mid x \in X\}.$$

We have $h^* = (\mathrm{cl}\, h)^* = (\overline{\mathrm{conv}} h)^*$. Moreover,

$$x^* \in \partial h(x) \Leftrightarrow h(x) + h^*(x^*) \leq \langle x, x^*\rangle \Leftrightarrow h(x) + h^*(x^*) = \langle x, x^*\rangle .$$

The *support* and the *indicator* functions of $A \neq \emptyset$ are respectively

$$\begin{aligned} \sigma_A(x^*) \quad &: \quad = \sup\{\langle a, x^*\rangle \mid a \in A\}, \text{ for } x^* \in X^*, \\ \mathrm{I}_A(x) \quad &: \quad = \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{if } x \in X \setminus A. \end{cases} \end{aligned}$$

σ_A is sublinear, lsc, and satisfies $\sigma_A = \sigma_{\overline{\mathrm{conv}} A} = \mathrm{I}^*_{\overline{\mathrm{conv}} A}$. Therefore, $\mathrm{epi}\, \sigma_A$ is a closed convex cone.

- For every family of functions $f_i, i \in I$, (I arbitrary), we have

$$(\inf_{i\in I} f_i)^* = \sup_{i\in I} f_i^*. \tag{4}$$

If $\{f_i, i \in I\} \subset \Gamma_0(X)$ and $\sup_{i\in I} f_i$ is proper, then

$$(\sup_{i\in I} f_i)^* = \mathrm{cl}\,\mathrm{conv}(\inf_{i\in I} f_i^*). \tag{5}$$

- For $f, g \in \Gamma_0(X)$ such that $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$, it is well known that

$$(f \Box g)^* = f^* + g^*,\ (f+g)^* = \mathrm{cl}(f^* \Box g^*). \tag{6}$$

Clearly, (6) and (4) imply that

$$\mathrm{epi}(f+g)^* = \mathrm{cl}(\mathrm{epi}\, f^* + \mathrm{epi}\, g^*),\ \mathrm{epi}(\sup_{i\in I} f_i)^* = \overline{\mathrm{conv}}(\cup_{i\in I}\, \mathrm{epi}\, f_i^*). \tag{7}$$

The closure operation in the first equation is superfluous if one of f and g is *continuous* at some point of $\mathrm{dom}\, f \cap \mathrm{dom}\, g$. Then, $\mathrm{epi}\, f^* + \mathrm{epi}\, g^*$ is w^*-closed (see, e.g., *Zalinescu'02*).

KKT'1 optimality conditions - FM property

Definition

We call *characteristic cone* of $\tau = \{f_t(x) \leq 0, \ t \in T; \ x \in \mathbb{C}\}$ to the convex cone

$$\mathbb{K} := \operatorname{cone}\left\{ \bigcup_{t \in T} \operatorname{epi} f_t^* \cup \operatorname{epi} \sigma_{\mathbb{C}} \right\} = \operatorname{cone}\left\{ \bigcup_{t \in T} \operatorname{epi} f_t^* \right\} + \operatorname{epi} \sigma_{\mathbb{C}}. \quad (8)$$

For the linear system (2),

$$\operatorname{epi} f_t^* = (a_t^*, b_t) + \mathbb{R}_+(\theta, 1), \ t \in T,$$

and

$$\operatorname{epi} \sigma_{\mathbb{C}} = \operatorname{epi} \sigma_X = \mathbb{R}_+(\theta, 1).$$

Hence,

$$\mathbb{K} = \operatorname{cone}\{(a_t^*, b_t), \ t \in T; \ (\theta, 1)\} \subset X^* \times \mathbb{R}. \quad (9)$$

Lemma

If $\mathbb{F} = \{x \in \mathbb{C} : \; f_t(x) \leq 0, \; t \in T\} \neq \emptyset$, *then*

$$\operatorname{epi} \sigma_{\mathbb{F}} = \operatorname{cl} \mathbb{K} = \overline{\operatorname{cone}} \left\{ \bigcup\nolimits_{t \in T} \operatorname{epi} f_t^* \cup \operatorname{epi} \sigma_{\mathbb{C}} \right\}.$$

Proof.

[Proof (sketch)] If $h := \sup\{f_t, \; t \in T; \; \mathrm{I}_{\mathbb{C}}\}$, we have

$$x \in \mathbb{F} \Leftrightarrow h(x) \leq 0 \Leftrightarrow h(x) = 0.$$

Then, by (5),

$$h^* = \{\sup\{f_t, \; t \in T; \; \mathrm{I}_{\mathbb{C}}\}\}^* = \operatorname{cl} \operatorname{conv} \left(\inf \{f_t^*, \; t \in T; \; \sigma_{\mathbb{C}}\}\right),$$

and

$$\operatorname{epi} \sigma_{\mathbb{F}} \stackrel{(*)}{=} \operatorname{cl}(\operatorname{cone} \operatorname{epi} h^*) = \operatorname{cl} \mathbb{K}.$$

$(*)$ follows from Lemma 3.1(b) in *Jey'03* (infinite-dimensional version) □

Theorem (generalized Farkas)

Let $\varphi, \psi \in \Gamma_0(X)$. *Then* $\varphi(x) \leq \psi(x)$ *for all* $x \in \mathbb{F}$, *assumed non-empty, if and only if*

$$\operatorname{epi} \varphi^* \subset \operatorname{cl}\left(\operatorname{epi} \psi^* + \mathbb{K}\right). \tag{10}$$

Proof.

$$\begin{aligned} \varphi(x) &\leq \psi(x)\ \forall x \in \mathbb{F} \iff \varphi \leq \psi + \mathrm{I}_{\mathbb{F}} \\ &\Leftrightarrow \quad (\psi + \mathrm{I}_{\mathbb{F}})^* \leq \varphi^* \\ &\Leftrightarrow \quad \operatorname{epi} \varphi^* \subset \operatorname{epi}(\psi + \mathrm{I}_{\mathbb{F}})^*, \end{aligned}$$

but applying (7), the previous lemma, and $\operatorname{cl}(A + B) = \operatorname{cl}(A + \operatorname{cl} B)$:

$$\begin{aligned} \operatorname{epi}(\psi + \mathrm{I}_{\mathbb{F}})^* &= \operatorname{cl}(\operatorname{epi} \psi^* + \operatorname{epi} \sigma_{\mathbb{F}}) \\ &= \operatorname{cl}(\operatorname{epi} \psi^* + \operatorname{cl} \mathbb{K}) = \operatorname{cl}(\operatorname{epi} \psi^* + \mathbb{K}). \end{aligned}$$

□

Corollary

Given $(a^*, \alpha) \in X^* \times \mathbb{R}$, *the inequality* $\langle a^*, x\rangle \leq \alpha$ *holds for all* $x \in \mathbb{F}$, *assumed non-empty (i.e.,* $\langle a^*, x\rangle \leq \alpha$ *is a continuous linear consequence of* τ*), if and only if*

$$(a^*, \alpha) \in \operatorname{cl} \mathbb{K}.$$

Proof.

Apply the generalized Farkas theorem with $\varphi = \langle a^*, \cdot\rangle - \alpha$ and $\psi \equiv 0$. Then, $\langle a^*, x\rangle \leq \alpha$ holds for all $x \in \mathbb{F}$ if and only if

$$\begin{aligned} (a^*, \alpha) + \mathbb{R}_+(\theta, 1) &= \operatorname{epi} \varphi^* \\ &\subset \operatorname{cl}(\operatorname{epi} \psi^* + \mathbb{K}) = \operatorname{cl}(\mathbb{R}_+(\theta, 1) + \mathbb{K}) = \operatorname{cl} \mathbb{K}. \end{aligned}$$

In other words, $\langle a^*, x\rangle \leq \alpha$ is a consequence of τ if and only if $(a^*, \alpha) \in \operatorname{cl} \mathbb{K}$. □

The following property is crucial in getting KKT-type optimality conditions for problem $(\mathcal{P})$.

Definition

We say that the *consistent* system $\tau = \{f_t(x) \leq 0,\ t \in T;\ x \in \mathbb{C}\})$ is *Farkas-Minkowski* (*FM*, in brief) if $\mathbb{K}$ is w^*-closed.

Theorem

If τ is FM, then every continuous linear consequence $\langle a^*, x\rangle \leq \alpha$ of τ, $(a^*, \alpha) \in X^* \times \mathbb{R}$, (i.e., $\langle a^*, x\rangle \leq \alpha$ holds for all $x \in \mathbb{F}$) is also consequence of a finite subsystem

$$\tau_S := \{f_t(x) \leq 0,\ t \in S;\ x \in \mathbb{C}\},\ \textit{with } S \subset T \textit{ and } |S| < \infty.$$

The converse statement holds if τ is linear.

The following theorem (*Dinh, Goberna, López, Son' 07*) provides *non-asymptotic* KKT-type optimality conditions for the problem

$$(\mathcal{P}) \quad \text{Min} \quad g(x)$$
$$\text{s.t.} \quad f_t(x) \leq 0, t \in T, \quad x \in \mathbb{C},$$

whose constraint system $\tau := \{f_t(x) \leq 0, \ t \in T; \ x \in \mathbb{C}\}$ has a non-empty set of feasible solutions $\mathbb{F}$.

Theorem (KKT'1)

Given the problem $(\mathcal{P})$*, assume that* τ *is FM and that* g *is continuous at some point of* $\mathbb{F}$*, and let* $\overline{x} \in \mathbb{F} \cap \operatorname{dom} g$*. Then* $\overline{x}$ *is a global minimum of* $(\mathcal{P})$ *if and only if there exists* $\lambda \in \mathbb{R}_+^{(T)}$ *such that* $\partial f_t(\overline{x}) \neq \emptyset, \forall t \in \operatorname{supp} \lambda,$ *and the KKT conditions*

$$\theta \in \partial g(\overline{x}) + \sum_{t \in T} \lambda_t \partial f_t(\overline{x}) + \mathrm{N}_{\mathbb{C}}(\overline{x}) \text{ and } \lambda_t f_t(\overline{x}) = 0, \ \forall t \in T, \qquad \text{(KKT'1)}$$

hold.

Here $\mathbb{R}_+^{(T)}$ is the space (convex cone) of functions $\lambda : T \to \mathbb{R}_+$ which vanishes at every point of T except at finitely many.

Proof.

[Proof of KKT'1 (sketch)] The point $\overline{x} \in \mathbb{F} \cap \operatorname{dom} g$ is a minimizer of $(\mathcal{P})$ if and only if

$$\theta \in \partial(g + \mathrm{I}_{\mathbb{F}})(\overline{x}) \overset{(*)}{=} \partial g(\overline{x}) + \partial \mathrm{I}_{\mathbb{F}}(\overline{x}) = \partial g(\overline{x}) + \mathrm{N}_{\mathbb{F}}(\overline{x}); \tag{11}$$

i.e., if and only if there exists $x^* \in \partial g(\overline{x})$ such that $\langle x^*, x\rangle \geq \langle x^*, \overline{x}\rangle$ is consequence of τ.

$(^*)$ Thanks to the continuity of g at some point of $\mathbb{F} \equiv \operatorname{dom} \mathrm{I}_{\mathbb{F}}$.

$(\Rightarrow)$ If $\overline{x}$ is a minimizer of $(\mathcal{P})$, since τ is FM we have

$$-(x^*, \langle x^*, \overline{x}\rangle) \in \operatorname{cl} \mathbb{K} = \mathbb{K} = \operatorname{cone}\left\{\bigcup_{t \in T} \operatorname{epi} f_t^*\right\} + \operatorname{epi} \sigma_{\mathbb{C}},$$

and $\exists\ \lambda \in \mathbb{R}_+^{(T)}$, $x_t^* \in \operatorname{dom} f_t^*$, $\alpha_t \geq 0$, $\forall t \in T$, $z^* \in \operatorname{dom} \sigma_{\mathbb{C}}$, $\beta \geq 0$, satisfying

$$-(x^*, \langle x^*, \overline{x}\rangle) = \sum_{t \in T} \lambda_t \left(x_t^*, f_t^*\left(x_t^*\right) + \alpha_t\right) + \left(z^*, \sigma_{\mathbb{C}}\left(z^*\right) + \beta\right),$$

leading to (KKT'1) by the relationship between the subdifferential and the conjugate.

$(\Leftarrow)$ Straightforward (standard argument). □

KKT'2 optimality conditions - LFM property

Let us introduce a weaker CQ. Given $z \in \mathbb{F}$, the set of indices corresponding to the *active constraints at* z is $T(z) := \{t \in T : \ f_t(z) = 0\}$. It is easily verified that

$$\mathrm{N}_{\mathbb{C}}(z) + \operatorname{cone}\left(\bigcup\nolimits_{t \in T(z)} \partial f_t(z)\right) \subseteq \mathrm{N}_{\mathbb{F}}(z). \tag{12}$$

Definition

The consistent constraint system τ is *locally Farkas-Minkowski* (*LFM*, in short) *at* $z \in \mathbb{F}$ if

$$\mathrm{N}_{\mathbb{F}}(z) \subseteq \mathrm{N}_{\mathbb{C}}(z) + \operatorname{cone}\left(\bigcup\nolimits_{t \in T(z)} \partial f_t(z)\right). \tag{13}$$

τ is said to be *LFM* if it is *LFM at every feasible point* $z \in \mathbb{F}$.

In LSIP ($\mathbb{C} = \mathbb{R}^n, \ f_t(x) = \langle a_t, x \rangle - b_t, \ t \in T$), (13) becomes

$$\mathrm{N}_{\mathbb{F}}(z) \subseteq \operatorname{cone}\{a_t, t \in T(z)\}.$$

The LFM property is closely related to the so-called *basic constraint qualification at* z. In fact, LFM and BCQ are equivalent under the continuity of the function $f := \sup_{t \in T} f_t$ at the reference point z and $z \in \operatorname{int} \mathbb{C}$.

The following proposition is a LFM counterpart of a similar property for FM systems.

Theorem

Let $z \in \mathbb{F}$. If τ is LFM at z and for certain $a^* \in X^*$ we have

$$\langle a^*, x\rangle \leq \langle a^*, z\rangle\,, \text{ for all } x \in \mathbb{F},$$

then $\langle a^*, x\rangle \leq \langle a^*, z\rangle$ is also a consequence of a finite subsystem of τ. The converse statement holds provided that τ is linear.

The converse statement in the last proposition does not hold in general for convex systems without any additional assumption.

Obviously,

$$\tau \text{ is } FM \Rightarrow \tau \text{ is } LFM \text{ at any } z \in \mathbb{F}.$$

The following theorem provides a *second* KKT-type optimality conditions for the problem

$$(\mathcal{P}) \quad \mathrm{Min}\, g(x) \quad \text{s.t. } f_t(x) \leq 0,\ t \in T, \quad x \in \mathbb{C}.$$

Theorem (KKT'2)

Given the problem $(\mathcal{P})$ and $\overline{x} \in \mathbb{F} \cap \operatorname{dom} g$, assume that τ is LFM at $\overline{x}$, and that g is continuous at some point of $\mathbb{F}$. Then $\overline{x}$ is a global minimum of $(\mathcal{P})$ if and only if there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that $\partial f_t(\overline{x}) \neq \emptyset$, $\forall t \in \operatorname{supp} \lambda$, and the KKT conditions hold

$$\theta \in \partial g(\overline{x}) + \sum_{t \in T} \lambda_t \partial f_t(\overline{x}) + \mathrm{N}_{\mathbb{C}}(\overline{x}) \text{ and } \lambda_t f_t(\overline{x}) = 0,\ \forall t \in T. \qquad \text{(KKT'2)}$$

Proof.

According to *Pshenichnyi-Rockafellar theorem* (e.g. *Zal'02 [Th. 2.9.1]*),

$$\begin{aligned} \overline{x} \text{ is optimal for } (\mathcal{P}) \quad &\Leftrightarrow \quad \partial g(\overline{x}) \cap (-\mathrm{N}_{\mathbb{F}}(\overline{x})) \neq \emptyset \\ &\Leftrightarrow \quad \theta \in \partial g(\overline{x}) + \mathrm{N}_{\mathbb{F}}(\overline{x}) \\ \overset{LFM}{\Leftrightarrow} \theta \quad &\in \quad \partial g(\overline{x}) + \sum_{t \in T(\overline{x})} \lambda_t \partial f_t(\overline{x}) + \mathrm{N}_{\mathbb{C}}(\overline{x}). \end{aligned}$$

KKT'3 asymptotic optimality conditions

Theorem (KKT'3)

Given the problem $(\mathcal{P})$, *let us assume that* X *is a* *Banach reflexive* *space, that* τ *is FM and* $(\operatorname{dom} g) \cap \operatorname{rint}(\mathbb{F}) \neq \emptyset$. *Then,* $\overline{x} \in (\operatorname{dom} g) \cap \mathbb{F}$ *is optimal for* $(\mathcal{P})$ *if and only if for each* $\varepsilon > 0$ *there exists* $\lambda^{\varepsilon} \in \mathbb{R}_{+}^{(T)}$ *such that* $\operatorname{supp} \lambda^{\varepsilon} \subset T(\overline{x})$ *and the following condition holds:*

$$\theta \in \partial_{\varepsilon} g(\overline{x}) + \sum\nolimits_{\operatorname{supp} \lambda^{\rho}} \lambda_t^{\varepsilon} \partial f_t(\overline{x}) + \mathrm{N}_{\mathbb{C}}(\overline{x}) + \varepsilon \mathbb{B}_{X^*}. \tag{14}$$

Proof.

[Sketch of the proof] $(\Rightarrow)$ Since $(\operatorname{dom} g) \cap \operatorname{rint}(\mathbb{F}) \neq \emptyset$, Cor. 5 in *Correa, Hantoute, López'16* yields

$$\partial(g + \mathrm{I}_{\mathbb{F}})(\overline{x}) = \bigcap\nolimits_{\varepsilon > 0} \operatorname{cl}(\partial g_{\varepsilon}(\overline{x}) + \mathrm{N}_{\mathbb{F}}(\overline{x})).$$

Then,

$$\overline{x} \text{ is optimal for } (\mathcal{P}) \Leftrightarrow \theta \in \bigcap\nolimits_{\varepsilon > 0} \operatorname{cl}(\partial g_{\varepsilon}(\overline{x}) + \mathrm{N}_{\mathbb{F}}(\overline{x})).$$

Proof.

[Sketch of the proof]
Since X is reflexive, $\mathrm{cl}(\partial g_\varepsilon(\overline{x}) + \mathrm{N}_{\mathbb{F}}(\overline{x}))$ coincides with the closure of $\partial g_\varepsilon(\overline{x}) + \mathrm{N}_{\mathbb{F}}(\overline{x})$ for the topology of the (dual) norm in X^* and, so, for every $\rho > 0$,

$$\theta \in \partial g_\varepsilon(\overline{x}) + \mathrm{N}_{\mathbb{F}}(\overline{x}) + \rho \mathbb{B}_{X^*}.$$

Thus, taking $\rho = \varepsilon$, there exists $a_\varepsilon^* \in \mathrm{N}_{\mathbb{F}}(\overline{x})$ such that

$$\theta \in \partial g_\varepsilon(\overline{x}) + a_\varepsilon^* + \varepsilon \mathbb{B}_{X^*}.$$

Since $a_\varepsilon^* \in \mathrm{N}_{\mathbb{F}}(\overline{x})$ is equivalent to say that $\langle a_\varepsilon^*, x\rangle \leq \langle a_\varepsilon^*, \overline{x}\rangle$ is a consequence of the FM system τ, we conclude the existence of $\lambda^\varepsilon \in \mathbb{R}_+^{(T)}$, $\operatorname{supp} \lambda^\varepsilon \subset T(\overline{x})$, such that

$$a_\varepsilon^* \in \sum_{\operatorname{supp} \lambda^\varepsilon} \lambda_t^\varepsilon \partial f_t(\overline{x}) + \mathrm{N}_{\mathbb{C}}(\overline{x}).$$

The necessity is proved.
$(\Leftarrow)$ Straightforward (standard arguments). □

Subdifferential calculus rules for the sum

- First results for the sum:

  a) Suppose that one of the following conditions hold:

  i) $X = \mathbb{R}^n$ and $\mathrm{rint}(\mathrm{dom}\, f) \cap \mathrm{rint}(\mathrm{dom}\, g) \neq \emptyset$,

  ii) X is a Hlcs and $(\mathrm{dom}\, g) \cap (\mathrm{cont}\, f) \neq \emptyset$ ($\mathrm{cont}\, f = \mathrm{int}(\mathrm{dom}\, f)$ if X is Banach and f is proper).

  Then

  $$\partial(f+g)(x) = \partial f(x) + \partial g(x).$$

  b) If $f, g \in \Gamma_0(X)$ one has (*Hiriart-Urruty, Phelps' 93*)

  $$\partial(f+g)(x) = \bigcap_{\varepsilon>0} \mathrm{cl}(\partial_\varepsilon f(x) + \partial_\varepsilon g(x)).$$

  c) If $(\mathrm{dom}\, g) \cap \mathrm{rint}(\mathrm{dom}\, f) \neq \emptyset$, and $f_{|\mathrm{aff}(\mathrm{dom}\, f)}$ is continuous on $\mathrm{rint}(\mathrm{dom}\, f)$ then (Th.12 in *Correa, Hantoute, López'16*) yields

  $$\partial(f+g)(x) = \bigcap_{\varepsilon>0} \mathrm{cl}(\partial f(x) + \partial g_\varepsilon(x)).$$

Subdifferential of the supremum function

- Let $f_t : X \to \mathbb{R} \cup \{+\infty\}$, $f_t \in \Gamma_0(X)$, $t \in T$, and $f := \sup_{t \in T} f_t$. Let $x \in X$ be such that for some $\varepsilon_0 > 0$,
  (i) $T_{\varepsilon_0}(x) := \{t \in T : \; f_t(x) \geq f(x) - \varepsilon_0\}$ is compact,
  (ii) $\forall z \in \operatorname{dom} f$, $t \mapsto f_t(z)$ is usc on $T_{\varepsilon_0}(x)$,
  Then (*Correa, Hantoute, López'19*)

$$\partial f(x) = \bigcap_{L \in \mathcal{F}(x)} \overline{\operatorname{conv}} \left(\bigcup_{t \in T(x)} \partial (f_t + \mathrm{I}_{L \cap \operatorname{dom} f})(x) \right),$$

  where $\mathcal{F}(x) := \{L \text{ is a finite-dimensional subspace of } X \text{ such that } x \in L\}$

- If f is *continuous at some point* ,

$$\partial f(x) = \overline{\operatorname{conv}} \left(\bigcup_{t \in T(x)} \partial f_t(x) \right) + \mathrm{N}_{\operatorname{dom} f}(x).$$

- If f is *continuous at* x,

$$\partial f(x) = \overline{\operatorname{conv}} \left(\bigcup_{t \in T(x)} \partial f_t(x) \right) \overset{X = \mathbb{R}^n}{=} \operatorname{conv} \left(\bigcup_{t \in T(x)} \partial f_t(x) \right).$$

KKT'4 conditions for SIP under compacity/continuity

Given the convex SIP problem $(\mathcal{P})$ we define

$$\mathbb{D} := \mathrm{dom}\, g \cap \mathrm{dom}(\sup_{t\in T} f_t).$$

Theorem

(Correa, Hantoute, López'19) Let $\bar{x}$ be a feasible point of $(\mathcal{P})$, with $T(\bar{x}) \neq \emptyset$, and assume that $\exists\ \varepsilon_0 > 0$ such that:
(i) the set $T_{\varepsilon_0}(\bar{x}) := \{t \in T \mid f_t(\bar{x}) \geq -\varepsilon_0\}$ is compact,
(ii) for each $z \in \mathbb{D} \cap \mathbb{C}$, the function $t \mapsto f_t(z)$ is usc on $T_{\varepsilon_0}(\bar{x})$.
Then, if $\bar{x}$ is optimal for $(\mathcal{P})$ the following conditions holds:

$$0_n \in \mathrm{co}\left\{\partial g(\bar{x}) \cup \bigcup_{t\in T(\bar{x})} \partial f_t(\bar{x})\right\} + \mathrm{N}_{\mathbb{D}\cap\mathbb{C}}(\bar{x}),$$

provided that

$$\mathrm{rint}(\mathrm{dom}\, f_t) \cap \mathrm{rint}(\mathbb{D} \cap \mathbb{C}) \neq \emptyset,\ \forall t \in T(\bar{x}),$$
$$\mathrm{rint}(\mathrm{dom}\, g) \cap \mathrm{rint}(\mathbb{D} \cap \mathbb{C}) \neq \emptyset.$$

Theorem

(Correa, Hantoute, López'19) Suppose that T is compact, for each $z \in \mathbb{D} \cap \mathbb{C}$ the function $t \mapsto f_t(z)$ is usc on T, and the family $\{\mathbb{C}, \operatorname{dom} f_t, t \in T, \operatorname{dom} g\}$ has the strong CHIP at $\bar{x}$. Then, if $\bar{x}$, with $T(\bar{x}) \neq \emptyset$, is optimal for $(\mathcal{P})$ the following inclusion holds

$$0_n \in \operatorname{conv}\left\{\partial g(\bar{x}) \cup \bigcup_{t \in T(\bar{x})} \partial f_t(\bar{x})\right\} + \mathrm{N}_C(\bar{x}) + \mathrm{N}_{\operatorname{dom} g}(\bar{x}) + \sum_{t \in T} \mathrm{N}_{\operatorname{dom} f_t}(\bar{x}).$$

If, additionally, the following Slater-type CQ holds

$$\sup_{t \in T} f_t(x_0) < 0 \textit{ for some } x_0 \in \mathbb{C} \cap \operatorname{dom} g,$$

then there exist a (possibly empty) finite set $\widehat{T}(\bar{x}) \subset T(\bar{x})$ such that $\partial f_t(\bar{x}) \neq \emptyset$ for $t \in \widehat{T}(\bar{x})$ and scalars $\lambda_t > 0$ for $t \in \widehat{T}(\bar{x})$, satisfying

$$0_n \in \partial g(\bar{x}) + \sum_{t \in \widehat{T}(\bar{x})} \lambda_t \partial f_t(\bar{x}) + \mathrm{N}_{\mathbb{C}}(\bar{x}) + \sum_{t \in T} \mathrm{N}_{\operatorname{dom} f_t}(\bar{x}), \tag{15}$$

with the convention that $\sum_{\emptyset} = \{0_n\}$.

Bibliographic comments

- The closedness of $\mathbb{K}$ was introduced in *Charnes, Cooper, Kortanek'65* as a general assumption for the duality theory in LSIP (see also *Goberna, López'98*).
- The FM property for convex systems was first studied in *Jeyakumar, Lee, Dinh'04*, with X being Banach and all the functions finite valued, under the name of *closed cone constraint qualification.* The FM property is strictly weaker than several known interior type reguality conditions.
- The LFM property, under the name of *basic constraint qualification* (*BCQ*), appeared in *Hiriart-Urruty, Lemaréchal'93*, relatively to the ordinary convex programming problem, with equality and inequality constraints.
- It was extended in *Puente, Vera de Serio'99* to the setting of linear semi-infinite systems. The consequences of its extension to convex semi-infinite systems were analyzed in *Fajardo, López'99.*
- For a deep analysis of BCQ and related conditions see also *Li, Nahak, Singer'00* and *Li, Ng'05.* An extensive comparative analysis of constraints qualifications for $(\mathcal{P})$ is also given in *Li, Ng, Pong'08*.

References

- CHARNES A. COOPER W.W., K.O. KORTANEK K.O., *On representations of semi-infinite programs which have no duality gaps*, Management Science 12 (1965) 113–121.
- CORREA R., HANOUTE A., LÓPEZ M.A., *Weaker conditions for subdifferential calculus of convex functions*. J. Funct. Anal. 271 (2016), no. 5, 1177–1212.
- CORREA R., HANOUTE A., LÓPEZ M.A., *Towards supremum-sum subdifferential calculus free of qualification conditions*. SIAM J. Optim. 26 (2016), no. 4, 2219–2234.
- CORREA R., HANOUTE A., LÓPEZ M.A., *Valadier-like formulas for the supremum function I*. J. Convex Anal. 25 (2018), 1253–1278.
- CORREA R., HANOUTE A., LÓPEZ M.A., *Valadier-like formulas for the supremum function II: the compactly indexed case*. J. Convex Anal. 26 (2019), 299–324.
- CORREA R., HANOUTE A., LÓPEZ M.A., *Moreau-Rockafellar-type formulas for the subdifferential of the supremum function*, SIAM J. Optim. 29 (2019).

Correa R., Hanoute A., LÓpez M.A., *Subdifferential of the supremum function via compactification of the index set*, Vietnam J. Math. to appear (2020)

Dinh N., Goberna M.A., López M.A., *From linear to convex systems: consistency, Farkas' lemma and applications*. J. Convex Anal. 13 (2006), 113–133.

Dinh N., Goberna M.A., López M.A., Son T.Q., *New Farkas-type constraint qualifications in convex infinite programming*, ESAIM: COCV 13 (2007)

Fajardo M.D., López M.A., *Locally Farkas-Minkowski systems in convex semi-infinite programming*. Journal of Optimization Theory and Applications 103 (1999), 313-335.

Goberna M.A., López M.A., *Linear Semi-infinite Optimization*. J. Wiley, Chichester (1998).

- Jeyakumar V., *Characterizing set containments involving infinite convex constraints and reverse-convex constraints*, SIAM J. Optimization 13 (2003) 947-959.
- Jeyakumar V., Lee G.M., Dinh N., *A new closed cone constraint qualification for convex Optimization,* Applied Mathematics Research Report AMR04/8, University of New South Wales, Sydney, Australia (2004)
- Hiriart-Urruty J.-B., Phelps R.R.: *Subdifferential calculus using epsilon-subdifferentials*, J. Funct. Anal., 118 (1993)
- Hiriart-Urruty J.-B., Lemaréchal C.: *Convex Analysis and Minimization Algorithms I, II,* Springer-Verlarg, Berlin, 1993.
- Hiriart-Urruty J.-B., Moussaoui M., Seeger A., Volle M.: *Subdifferential calculus without qualification conditions, using approximate subdifferentials: A survey,* Nonlinear Anal., 24 (1995)

- Li W., Nahak C., Singer I., *Constraint qualification for semi-infinite systems of convex inequalities*. SIAM J. Optimization 11 (2000) 31-52.
- Li Ch., Ng K.F., *On constraint qualification for an infinite system of convex inequalities in a Banach space*. SIAM J. Optimization 15 (2005) 488-512.
- Li Ch., Ng K.F., Pong T.K., *Constraint qualifications for convex inequality systems with applications in constrained optimization*. SIAM J. Optim. 19 (2008), 163–187.
- Puente R., Vera de Serio V.N., *Locally Farkas-Minkowski linear semi-infinite systems*. *TOP* 7 (1999) 103-121.
- Thibault, L., *Sequential convex subdifferential calculus and sequential Lagrange multipliers*, SIAM J. Contr. Optim. 35 (1997)
- Zalinescu, C. *Convex analysis in general vector spaces*. World Scientific Publishing Co., Inc., River Edge, NJ, 2002.

Thanks you for your attention