Old and new results on equilibrium and quasi- equilibrium problems

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Outline

J. Cotrina M.T, J. Zúñiga ***An existence result for quasi-equilibrium problems via Ekeland's variational principle,*** *https://hal.archives-ouvertes.fr/hal-02650953 [1]*

Outline

Let C be a nonempty subset of a topological space X and $f : C \times C \to \mathbb{R}$, be a given bifunction.

Equilibrium problem:

$$\textit{Find} \quad x \in C \quad \textit{such that} \quad f(x, y) \geq 0, \ \textit{for all} \ y \in C.$$

Existence results go back to

- the celebrated Knaster-Kuratowski-Mazurkiewicz (1929)
- and its generalization in finite dimension by Ky-Fan (1961)
- and its extension by Brézis- Nirenberg-Stampacchia to a subset which is not necessarily compact by using a ” coercivity-type condition"

Nowadays a great interest is given to the convergence analysis of iteratives methods to reach solutions of equilibrium problems.

Theorem (Fan 's Minimax Principle 72)

Assume C is a compact and convex subset of a Hausdorff topological vector space and suppose that $f : C \times C \to \mathbb{R}$ satisfies conditions :

(1) $f(x,x) \leq 0$ *for all* $x \in C$;

(2) *For every* $x \in C$ *the set* $\{y \in C : f(x,y) > 0\}$ *is convex;*

(3) *for every* y *fixed in* C, $f(\cdot, y)$ *is lower semicontinuous on* C.

Then, there exists $x_0 \in C$ *such that* $f(x_0, y) \leq 0$ *for every* $y \in C$.

With the goal to apply to variational inequalities ($f(x,y) = \langle Ax, x-y\rangle$ with A **monotone or pseudo-monotone**) Fan's result was extended by Brézis-Niremberg and Stampacchia

Theorem (Brézis-Niremberg-Stampacchia 1972)

They supposed (1), (2) and

(3′) *For every fixed* $y \in C$, $f(\cdot, y)$ *is lower semicontinuous on the intersection of* C *with any finite dimensional subspace of* X;

(4) *Whenever* $x, y \in C$ *and* (x_i) *is a net on* C *converging to* x, *then the following implication holds:*

$$f(x_i,(1-t)x+ty) \le 0 \quad \textit{for every} \quad t \in [0,1] \Longrightarrow \quad f(x,y) \le 0;$$

(5) *There is a compact subset* L *of* X *and* $y_0 \in L \cap C$ *such that* $f(x, y_0) > 0$ *for* $x \in C, y_0 \notin L$.

Then there is $x_0 \in L \cap C$ *such that* $f(x_0, y) \le 0$ *for all* $y \in C$. *In particular*

$$\inf_{x\in C}\sup_{y\in C} f(x,y) \le 0.$$

The proof is based on a result from Knaster- Kuratowski-Mazurkiewicz and BNS implies the Von Neuman-Sion minimax principle and the Hartman-Stampacchia theorem.

- Mosco introduced implicit variational problems nowadays known as quasi-equilibrium problem
- Blum & Oettli proposed a mixed version of Ky Fan's minimax inequality by perturbing the monotone objective bifunction f by a nonmonotone term. This covers further problems such as the nonconvex and nonsmooth Panagiotopoulos inequalties considered by Chadli, Chbani and Riahi
- Ait Mansour and Riahi studied for example the quantitative stability of equilibrium problems
- relaxed generalized quasimonotonicity and quasiconvexity, for example see Castellani & M. Giuli, Hadjisawas, Bianchi-Kassai-Pini, etc..
- iterative methods for (EP): for example Moudafi extended the Proximal point algorithm (PPA) from variational inequalities to equilibrium problems
- and many other authors.

Outline

Let C be a non-empty subset of $\mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}$ and $T : \mathbb{R}^n \to \mathbb{R}^n$.

1. The *minimization problem* **MIN(**h,C) associated to h and C consists to find $x \in C$ such that
$$h(x) \le h(y), \text{ for all } y \in C.$$
Use $f(x,y) = h(x) - h(y)$

$$\bar{x} \quad \textit{solves} \quad \textbf{\textit{MIN(}}h,C) \quad \Longleftrightarrow \quad \bar{x} \quad \textit{solves} \quad \textbf{\textit{EP(}}f,C)$$

2. The *Stampacchia variational inequality problem* **SVI(T,C)** associated to T and C consists to find $x \in C$ such that

SVI(T,C) Find $\bar{x} \in C, \bar{\xi} \in T(x)$ such that $\langle \bar{\xi}, y - \bar{x} \rangle \ge 0$ for all $y \in C.$

Take :
$$f(x,y) = \max_{\xi \in T(x)} \langle \xi, y - x \rangle.$$

$$\bar{x} \quad \textit{solves} \quad \textbf{\textit{SVI(T,C)}} \quad \Longleftrightarrow \quad \bar{x} \quad \textit{solves} \quad \textbf{\textit{EP(}}f,C)$$

3. Let $\mathbb{H} = \mathbb{H}^\star$ (Hilbert). Let $T : \mathbb{H} \rightrightarrows X^\star$. be such that $T(x) \neq \emptyset$, $T(x) \subset C \subset \mathbb{H}$.

(FP) Find $\bar{x} \in C$ such that $\bar{x} \in T(\bar{x}).$

Take $f(x,y) = \max_{\xi \in T(x)} \langle x - \xi, y - x \rangle.$

$$\bar{x} \quad \textit{solves} \quad \textbf{\textit{FP}} \quad \Longleftrightarrow \quad \bar{x} \quad \textit{solves} \quad \textbf{\textit{EP(}}f,C)$$

- **Saddle point problem**

  $\mathbb{H}_1$ and $\mathbb{H}_2$ are 2 Hilbert spaces; $C_1 \subset \mathbb{H}_1$ and $C_2 \subset \mathbb{H}_2$ 2 closed convex subsets.

  (x_1, x_2) is a saddle point of $\Theta : C_1 \times C_2 \to \mathbb{R}$ if:

$$\Theta(x_1, v) \leq \Theta(u, x_2) \qquad \text{for all} \quad (u, v) \in C_1 \times C_2.$$

  Set $C = C_1 \times C_2$ anf $f : C \times C :\to R$ given by

$$f(x_1, x_2), (y_1, y_2)) = \Theta(y_1, x_2) - \Theta(x_1, y_2).$$

Then, the equilibrium problem associated to f is ***equivalent*** *to the saddle point problem associated to Θ.*

- **Nash equilibrium problem**

  Given a finite index set $I = \{1, 2, ..., N\}$,. Every player i has a strategy set C_i and $C = \prod_{i \in I} C_i$ be the product of the strategy sets.

  For each $i \in I$, let $f_i : C \to \mathbb{R}$ be the loss function of the i-th player which depends on the strategies of all players.

  Set $f : C \to \mathbb{R}$ defined by $f(x) = (f_1(x), f_2(x), ..., f_N(x))$

  $C_{-i} = \prod_{j \neq i, j=1}^{N} C_j$ and $x_{-i} = (x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_N)$.

  For each x, y, set

  $$x(y_i) = (x_1, x_2, ..., x_{j-1}, y_i, X_{j+1} ..., x_N).$$

  The point $\bar{x} \in C$ is a Nash equilibrium if for all $i \in I$ there holds

  $$f_i(\bar{x}) \leq f_i(\bar{x}(y_i)), \forall y \in C_i, \forall i$$

  Take: $f(x, y) = \sum_{i \in I} (f_i(x(y_i)) - f_i(x))$.

  $$\bar{x} \text{ is a Nash equilibrium} \iff \bar{x} \text{ solves EP}(f, C)$$

We have a Nash Equilibrium when player can reduce his loss by varying his strategy alone

Outline

Tian & Zhou [JME 95] were interested in the question to know under which minimal conditions

- a function attains its maximum on a compact set;
- the set of maximum point of a function on a compact set is nonempty;
- the maximum correspondance is closed.

- $h : C \to \mathbb{R} \cup \{+\infty\}$ is transfer lower continuous (tlc, for short) if, for each $x, y \in C$ such that $h(x) > h(y)$, there exist $y' \in C$ and V_x a neighbourhood of x such that $h(x') > h(y')$, for all $x' \in V_x \cap C$.
- $h : C \to \mathbb{R} \cup \{+\infty\}$ is transfer weakly lower continuous (tlc, for short) if, for each $x, y \in C$ such that $h(x) > h(y)$, there exist $y' \in C$ and V_x a neighbourhood of x such that $h(x') \geq h(y')$, for all $x' \in V_x \cap C$.

Results

Theorem (Weierstrass-type theorem Tian & Zhou [JME 95)

] Let C be a compact subset of a topological space X and $f : C \to \mathbb{R} \cup \{+\infty\}$ be given. Then,

f *attains its minimum on* $C \iff f$ *is transfer weakly tlc.*

Theorem ()

Let C be a compact subset of a topological space X and $f : C \to \mathbb{R} \cup \{+\infty\}$ be given. Then,

$$\min_C f \quad \text{is nonempty and compact} \quad \iff \quad f \quad \text{is weakly tlc.}$$

(Fact)

$$h \quad \text{is tlc} \quad \iff \quad \bigcap_{x \in C} S_h(h(x)) = \bigcap_{x \in C} \overline{S_h(h(x))},$$

Contrary to lower semicontinuity, tlc is **not stable under addition**:

Example (Ex. 1)

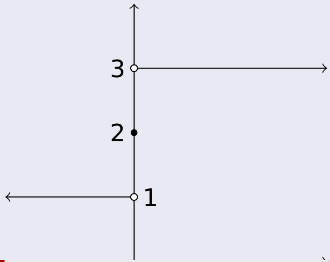
Take

$$h(x) := \begin{cases} x+1, & x<0 \\ 2, & x=0 \\ x+3, & x>0 \end{cases} \quad \textit{and} \quad g(x) := -x.$$

*h and g are tlc. However, $h+g$ **fails** to be tlc at* 0.

$$(h+g)(x) = \begin{cases} 1, & x<0 \\ 2, & x=0 \\ 3, & x>0 \end{cases}$$

see the graph represented below:



Given a nonempty subset C of a topological space X, every function $h : C \to \mathbb{R}$ (not necessarily lsc) the largest lower semicontinuous minorant $\bar{h}$ of h is called the *lower semicontinuous regularization* of h.

$$\operatorname{Epi} \overline{h} := \overline{\operatorname{Epi} h}$$

$$\Updownarrow$$

$$\bar{h}(x) = \liminf_{y \to x} h(y) = \sup_{U} \inf_{y \in U} h(y)$$

It is well known that for any $x \in C$ and any $\lambda \in \mathbb{R}$

1. $\overline{h}(x) = \inf\{\lambda \in \mathbb{R} : \ x \in \overline{S_\lambda(h)}\}$;
2. $\overline{h}(x) \leq h(x)$;
3. $\overline{S_h(x)} \subset S_{\overline{h}}(x)$;
4. $S_{\overline{h}}(\lambda) = \bigcap_{\mu > \lambda} \overline{S_h(\mu)}$;

A lower semicontinuous regularization is well-defined if $\overline{h}(x) > -\infty$, for all $x \in C$.

In other words, if h admits a lsc minorant.

Proposition (⌂)

Let C be a nonempty subset of a topological space X and $h : C \to \mathbb{R}$ be a function.

If h is tlc, then its lower semicontinuous regularization $\bar{h}$ is well-defined.

Proof.

It is enough to consider the case when h is not bounded below. Then, for each $x \in C$ there exists $y \in C$ such that $h(x) > h(y)$. Since h is tlc there exist V_x a neighbourhood of x and $y' \in C$ such that $h(x') > h(y')$, for all $x' \in V_x$, which in turn implies $\overline{h}(x) \geq h(y')$. Therefore, $\overline{h}$ is well-defined. □

Proposition (▲)

Let C be a nonempty subset of a topological space X and $h : C \to \mathbb{R}$ be a function. Then, the following holds

$$\inf_{x \in C} h(x) = \inf_{x \in C} \overline{h}(x).$$

Proposition (◐◐)

Let C be a nonempty subset of a topological space X and $h : C \to \mathbb{R}$ be a function. Then

$$\operatorname{argmin}_C h = \operatorname{argmin}_C \overline{h} \quad \Longleftrightarrow \quad h \quad \text{is tlc.}$$

Transfer lower continuity is essential in the previous result.

Example (Ex. 3)

Let

$$h(x) := \begin{cases} |x|, & x \neq 0 \\ 1, & x = 1 \end{cases}$$

Clearly $\overline{h}(x) = |x|$. *shows the graphs of* h *and* $\overline{h}$*, respectively.*

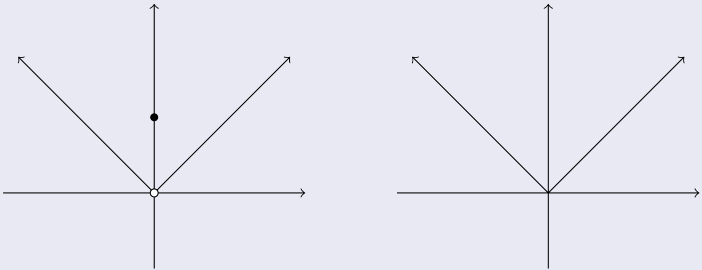


Figure: *graphs of* h *and* $\overline{h}$

Moreover

$$\operatorname{argmin} h = \emptyset \quad \textit{and} \quad \operatorname{argmin} \overline{h} = \{0\}.$$

Proposition (✠)

Let C be a nonempty subset of a topological space X and $h : C \to \mathbb{R}$ be a tlc function. If there exists $x \in C$ such that

$$\overline{h}(x) \leq h(y), \quad \textit{for all } y \in C;$$

then $x \in \underset{C}{\operatorname{argmin}}\, h$.

Proof.

Since h is tlc, its lower semicontinuous regularization $\overline{h}$ is well-defined, due to Proposition (✠✠). Clearly, $\operatorname{Epi}(h) \subset C \times \left[\overline{h}(x), +\infty\right[$. Thus, we deduce that $x \in \underset{C}{\operatorname{argmin}}\, \overline{h}$. The result follows from Proposition [◐◐] . □

Oettli & Th (1993), Bianchi et al. (2005), Alleche & Radulescu (2015).,...
The approach given in all these results is based on the assumption that the equilibrium bifunction should satisfy the triangle inequality property:

$$f(x,y) \leq f(x,z) + f(z,y), \forall x,y,z \in C.$$

Theorem (Castellani & Giuli)

Let C be a nonempty closed subset of a complete metric space (X,d) and $f : C \times C \to \mathbb{R}$ be a bifunction. Assume that there exists a function $h : C \to \mathbb{R}$, bounded from below and lower semicontinuous such that

$$f(x,y) \geq h(y) - h(x), \ \forall \ x,y \in C.$$

Then, for every $\varepsilon > 0$ and for every $x_0 \in C$ there exists $\overline{x} \in C$ such that

(a) $h(\overline{x}) \leq h(x_0) - \varepsilon d(x_0, \overline{x})$;

(b) $f(\overline{x}, y) + \varepsilon d(\overline{x}, y) > 0$, *for every* $y \in C \setminus \{\overline{x}\}$.

(Remark)

$f(x,y) = \phi(x) - \phi(y)$ *gives the original Ekeland Theorem*

Let us restate **EVP** in terms of lower semicontinuous regularizations.

Theorem (▲)

Let C be a nonempty closed subset of the complete metric space (X, d), and $h : C \to \mathbb{R}$ be a function bounded from below. For every $\varepsilon > 0$, and for any $x_0 \in C$, there exists $\hat{x} \in C$ such that

$$\overline{h}(\hat{x}) + \varepsilon d(x_0, \hat{x}) \leq h(x_0), \text{ and}$$

$$h(x) + \varepsilon d(x, \hat{x}) > \overline{h}(\hat{x}), \text{ for all } x \in C \setminus \{\hat{x}\}.$$

Theorem (☎)

*Theorems **EVP** through Theorem (▲) are equivalent.*

As a direct consequence of Theorem (☼☼) we have the following corollary.

Corollary (△△)

Let C be a nonempty closed subset of a complete metric space (X, d) and $f : C \times C \to \mathbb{R}$ be a bifunction. Assume that there exists a bifunction $g : C \times C \to \mathbb{R}$ such that:

1. $f \geq g$;
2. *g is bounded from below and lsc with respect to its second argument;*
3. *g vanishes on the diagonal of $C \times C$;*
4. *g satisfies the triangle inequality property.*

Then, for all $\varepsilon > 0$, and all $x_0 \in C$, there exists $\hat{x} \in C$ such that

$$g(x_0, \hat{x}) + \varepsilon d(x_0, \hat{x}) \leq 0$$

$$f(\hat{x}, x) + \varepsilon d(x, \hat{x}) > 0 \text{ for every } x \in C \setminus \{\hat{x}\}.$$

Some definitions related to bifunctions

A bifunction $f : C \times C \to \mathbb{R}$ is said:

- to have the *triangle inequality property* on C if, for all $x, y, z \in C$ the following holds
$$f(x, y) \leq f(x, z) + f(z, y);$$
- to be *cyclically monotone* on C [Bianchi, Kasay and Pini [2] (2005)] if, for all $n \in \mathbb{N}$ and all $x_0, x_1, \ldots, x_n \in C$ the following holds ,
$$\sum_{i=0}^{n} f(x_i, x_{i+1}) \leq 0,$$
with $x_{n+1} = x_0$; Extends Rockafellar's classical definition given for set-valued mappings [3].
- to be *monotone* on C if, for all $x, y \in C$ the following holds
$$f(x, y) + f(y, x) \leq 0;$$
- to be *pseudo-monotone* on C if, for all $x, y \in C$ the following implication holds
$$f(x, y) \geq 0 \implies f(y, x) \leq 0.$$

Example (Hadjisavvas & Khatibzadeh -2007-)

Let C be a nonempty subset of a topological space X and let $h : C \to \mathbb{R}$ be a function with a well-defined lsc regularization. Consider

$$g(x,y) := h(y) - h(x) \text{ and } f(x,y) := \overline{h}(y) - h(x). \tag{1}$$

f is cyclically monotone and g satisfies the triangular inequality property. Moreover, $g(x,y) \geq f(x,y)$ for all $x, y \in C$.

Take $f : C \times C \to \mathbb{R}$, and use $\hat{f} : C \times C \to \mathbb{R}$ by $\hat{f}(x,y) := -f(y,x)$.

If f verifies the triangle inequality property, then $\hat{f}$ is cyclically monotone.

Cyclic monotonicity of $\hat{f}$ is equivalent to the existence of a function $h : C \to \mathbb{R}$ such that

$$\hat{f}(x,y) \leq h(y) - h(x), \ \forall x, y \in C.$$

Moreover,

$$f(x,y) \geq h(y) - h(x) \geq \hat{f}(x,y).$$

In addition, f monotone $\Longrightarrow$ f(x,y)=h(y)-h(x), $\forall x, y \in C$ $\Longrightarrow$ f cyclically monotone.

Outline

Let C be a nonempty subset of a topological space X and $f : C \times C \to \mathbb{R}$, be a given bifunction. We denote by **EP**(f, C) *the solution set* of the *equilibrium problem*:

$$\text{Find} \quad x \in C \quad \text{such that} \quad f(x, y) \geq 0, \ \text{for all } y \in C. \tag{2}$$

In a similar way, **MEP**(f, C) denotes the *solution set* of the so-called *Minty equilibrium problem*:

$$\text{Find} \quad x \in C \quad \text{such that} \quad f(y, x) \leq 0, \ \text{for all } y \in C. \tag{3}$$

II $\hat{f}(x, y) := -f(y, x)$, then

$$\mathbf{EP}(f, C) = \mathbf{MEP}(\hat{f}, C) \ \text{ and } \ \mathbf{EP}(\hat{f}, C) = \mathbf{MEP}(f, C).$$

Provided that $f(x, y) \geq h(y) - h(x)$, which implies that $\hat{f}$ is cyclically monotone, we may observe that

$$\mathbf{MEP}(f, C) \subset \operatorname*{argmin}_C h \subset \mathbf{EP}(f, C). \tag{4}$$

Moreover, if f is pseudo-monotone, then the above inclusions are actually equalities.

Remark

If the bifunction f vanishes on the diagonal of $C \times C$, then

$$x \in \boldsymbol{EP}(f, C) \Leftrightarrow x \in \underset{C}{\operatorname{argmin}} f(x, \cdot) \ \ \textit{and} \ \ x \in \boldsymbol{MEP}(f, C) \Leftrightarrow x \in \underset{C}{\operatorname{argmin}} \hat{f}(x, \cdot).$$

Moreover,

$$\boldsymbol{EP}(f, C) \subset \bigcup_{x \in C} \underset{C}{\operatorname{argmin}} f(x, \cdot) \ \ \textit{and} \ \ \boldsymbol{MEP}(f, C) \subset \bigcup_{y \in C} \underset{C}{\operatorname{argmin}} \hat{f}(y, \cdot).$$

Theorem (☼)

Let C be a compact and nonempty subset of a topological space X, and $f : C \times C \to \mathbb{R}$ be a bifunction. If there exists a tlc function $h : C \to \mathbb{R}$ with

$$f(x, y) \geq h(y) - h(x), \ \textit{for all } x, y \in C;$$

then, the set $\boldsymbol{EP}(f, C)$ is nonempty.

Given C a subset of $\mathbb{R}^n$, a set-valued map $K : C \rightrightarrows C$ and a function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, we consider the quasi-equilibrium problem (**QEP**) associated to f and K:

$$\text{Find } x \in K(x) \text{ such that } f(x,y) \geq 0, \text{ for all } y \in K(x) \tag{QEP}$$

In a similar way, $\mathbf{MEP}(f,C)$ denotes the *solution set* of the so-called *Minty equilibrium problem*

$$\text{Find} \quad x \in C \quad \text{such that} \quad f(y,x) \leq 0, \text{ for all } y \in C. \tag{5}$$

$$\hat{f}(x,y) := -f(y,x).$$

Clearly, they satisfy

$$\mathbf{EP}(f,C) = \mathbf{MEP}(\hat{f},C) \text{ and } \mathbf{EP}(\hat{f},C) = \mathbf{MEP}(f,C).$$

Provided that $f(x,y) \geq h(y) - h(x)$, which implies that $\hat{f}$ is cyclically monotone, we may observe that

$$\mathbf{MEP}(f,C) \subset \operatorname*{argmin}_C h \subset \mathbf{EP}(f,C). \tag{6}$$

Moreover, if f is pseudo-monotone, then the above inclusions are actually equalities.

Remark

If the bifunction f vanishes on the diagonal of $C \times C$, then

$$x \in \textbf{EP}(f, C) \Longleftrightarrow x \in \operatorname*{argmin}_C f(x, \cdot) \; \textit{and} \; x \in \textbf{MEP}(f, C) \Longleftrightarrow x \in \operatorname*{argmin}_C \hat{f}(x, \cdot).$$

Moreover,

$$\textbf{EP}(f, C) \subset \bigcup_{x \in C} \operatorname*{argmin}_C f(x, \cdot) \;\; \textit{and} \;\; \textbf{MEP}(f, C) \subset \bigcup_{y \in C} \operatorname*{argmin}_C \hat{f}(y, \cdot).$$

Theorem (☺)

Let C be a compact subset of a topological space X, and $f : C \times C \to \mathbb{R}$ be a bifunction. If there exists a tlc function $h : C \to \mathbb{R}$ with

$$f(x,y) \geq h(y) - h(x), \text{ for all } x, y \in C;$$

then, the set $\boldsymbol{EP}(f, C)$ *is nonempty.*

Proof.

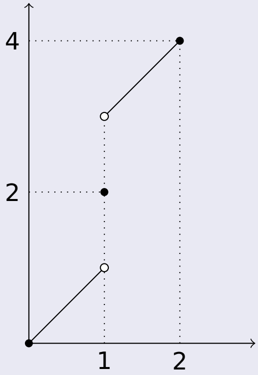
From Theorem (🌥) the set $\operatorname{argmin}_C h$ is nonempty. The result follows from (6). □

The previous result was given in Castellini & Guili 2016 Theorem 3.11, but instead of considering the transfer lower continuity of h, the authors assumed lower semicontinuity.

Example (Ex. 5)

Take

$$h(x) := \begin{cases} x, & 0 \leq x < 1 \\ 2, & x = 1 \\ x + 2, & 1 < x \leq 2 \end{cases}$$



As a direct consequence we have the following corollary, which is generalization of Giuli 2017, Theorem 3.1.

Corollary (△ △ △)

Let C be a compact subset of a topological space X, and $f : C \times C \to \mathbb{R}$ be a bifunction. If there exists a tlc function $h : C \to \mathbb{R}$ with

$$f(x,y) \leq h(y) - h(x), \text{ for all } x,y \in C;$$

then, the set $\boldsymbol{MEP}(f,C)$ *is nonempty.*

Outline

Given a nonempty subset C of a complete metric space (X, d), a bifunction $f : C \times C \to \mathbb{R}$ and a set-valued mapping $K : C \rightrightarrows C$, we denote by $\mathbf{QEP}(f, K)$ the solution set of the so-called *quasi-equilibrium problem*:

$$\text{Find} \quad x \in C \quad \text{such that} \quad x \in K(x) \ \wedge \ f(x, y) \geq 0, \ \text{for all } y \in K(x). \tag{7}$$

Lemma (◑)

Let C be a nonempty subset of a complete metric space (X, d), $K : C \rightrightarrows C$ be a set-valued mapping and $h : C \to \mathbb{R}$ be a function bounded from below. We assume that for every $\varepsilon > 0$, and for any $x_0 \in C$ the following implication holds:

$$\overline{h}(x) + \varepsilon d(x, x_0) \leq h(x_0) \implies \exists y \in K(x), \ h(y) + \varepsilon d(x, y) \leq \overline{h}(x).$$

Then, there exists $\hat{x} \in$ $\mathbf{Fix}(K)$ satisfying

$$\overline{h}(\hat{x}) + \varepsilon d(x_0, \hat{x}) \leq h(x_0),$$

and

$$h(x) + \varepsilon d(x, \hat{x}) > \overline{h}(\hat{x}), \text{ for all } x \in C \setminus \{\hat{x}\}.$$

Associated to f and K, we notice that if there exists a function $h : C \to \mathbb{R}$ such that $f(x, y) \geq h(y) - h(x)$ (in other words, $\hat{f}$ is cyclically monotone), then

$$\operatorname*{argmin}_C h \cap \mathbf{Fix}(K) \subset \mathbf{QEP}(f, K). \tag{8}$$

Theorem (❄)

Let C be a nonempty closed subset of a complete metric space (X, d), let $K : C \rightrightarrows C$ be a set-valued mapping, and let $f : C \times C \to \mathbb{R}$ be a bifunction. Assume that the following conditions hold.

1. ***Fix**(K) is compact and nonempty;*
2. *there exists a tsc function bounded below $h : C \to \mathbb{R}$ such that*

$$f(x, y) \geq h(y) - h(x), \quad \text{for all } x, y \in C.$$

Suppose that for each $\varepsilon > 0$ and each $x_0 \in X$ the following implication holds:

$$\overline{h}(x) + \varepsilon d(x, x_0) \leq h(x_0) \implies \exists y \in K(x),\ h(y) + \varepsilon d(x, y) \leq \overline{h}(x).$$

*Then, the set **QEP**(f, K) is nonempty.*

The previous result is related to Castellini & Guili 2016 but instead of transfer lower continuity, the authors assumed lower semi-continuity.

As a direct consequence of Theorem (❋) we derive.

Corollary (☢)

Let C be a nonempty closed subset of a complete metric space (X, d), $K : C \rightrightarrows C$ be a set-valued mapping, and let $h : C \to \mathbb{R}$ be a function. Assume that the following conditions hold.

1. ***Fix**(K) is compact and nonempty;*
2. *h is a tlc function bounded from below.*

Suppose that for each $\varepsilon > 0$, and each $x_0 \in X$ the following implication holds:

$$\overline{h}(x) + \varepsilon d(x, x_0) \leq h(x_0) \implies \exists y \in K(x),\ h(y) + \varepsilon d(x, y) \leq \overline{h}(x).$$

*Then, there exists $\hat{x} \in$ **Fix**(K) such that*

$$h(\hat{x}) \leq h(x),\ \text{for all } x \in K(\hat{x}).$$

Let I be an index set. For each $i \in I$, we consider a complete metric space (X_i, d_i), a nonempty closed subset C_i of X_i and a set-valued mapping $K_i : C_i \rightrightarrows C_i$. We define the set-valued mapping $K : C \rightrightarrows C$ by

$$K(x) := \prod_{i \in I} K_i(x_i),$$

where $C = \prod C_i$ and $x = (x^i)_{i \in I}$. By a *system of quasi-equilibrium problems* we understand the problem of finding

$$(\blacklozenge) \qquad \hat{x} \in \mathbf{Fix}(K) \text{ such that } f_i(\hat{x}, y^i) \geq 0 \text{ for all } y \in K(\hat{x}), \tag{9}$$

where $f_i : C \times C_i \to \mathbb{R}$.
It is important to observe that

$$\mathbf{Fix}(K) = \prod_{i \in I} \mathbf{Fix}(K_i).$$

The following result generalizes Castellani-Giuli 2016 (Theorem 4.2], Alech-Radulescu 2015 (Proposition 4.2) and Bianchi-Kassai-Pini 2005 (Proposition 2)

Theorem (☁)

For each $i \in I$, let C_i be a nonempty closed subset of a topological space X_i, and let $f_i : C \times C_i \to \mathbb{R}$ be a bifunction such that

$$(\clubsuit) \qquad f_i(x, y^i) \geq h_i(y^i) - h_i(x^i), \ \forall x, y \in C \tag{10}$$

holds for some transfer lower continuous function $h_i : C_i \to \mathbb{R}$ that is also bounded from below. Then, the system of equilibrium problems admits at least a solution.

Remark

Condition (♣) *is equivalent to the following: for any* $x_1, x_2, \ldots, x_m \in C$ *it holds*

$$(\spadesuit) \qquad \sum_{j=1}^{m} f_i(x_j, x^i_{j+1}) \geq 0 \tag{11}$$

where $x_{m+1} = x_1$. *It follows from the same steps of the proof of [5, Proposition 5.1].*

We denote by **SEP**(f_i, C_i, I) the solution set of (9), when $K_i(x^i) = C_i$, for all $x^i \in C_i$. If I is a finite index set, as a particular case, we define the bifunction $f : C \times C \to \mathbb{R}$ by

$$(\text{▲}) \quad f(x,y) := \sum_{i \in I} f_i(x, y^i). \tag{12}$$

Proposition (◐)

Assume that I *is a finite index set. Then* **SEP**$(f_i, C_i, I) \subset$ **EP**(f, C). *The equality holds provided that* $f_i(x, x^i) = 0$, *for all* $i \in I$.

Remark

Assume that I is a finite index set. If for each $i \in I$ the function f_i satisfies condition **(♠)** *then the function $\hat{f}$ is cyclically monotone, where f is defined as (⛺).*

Given a finite index set I and for each $i \in I$, we consider a subset C_i of a topological space and a function $f_i : C \times C_i \to \mathbb{R}$. We say that the family of functions $\{f_i\}_{i \in I}$ have the *transfer lower continuity property* if there exists a tlc function $h : C \to \mathbb{R}$ such that the bifunction f defined by (⛺) satisfies

$$f(x,y) \geq h(y) - h(x).$$

Remark

A few remarks are needed.

1. *Let f be associated to a family of functions with the transfer lower continuity property and defined in (⛺). Then, the bifunction $\hat{f}$ is cyclically monotone;*
2. *If for each $i \in I$, the function f_i is usc in its second argument; then the familty of function $\{f_i\}_{i \in I}$ has the transfer lower continuity property. It is due to Castellani & Giuli Theorem 2.16.*

Theorem (⛰)

Assume that I is a finite index set and the family of functions $\{f_i\}_{i\in I}$ has the transfer lower continuity property. If $f_i(x, x^i) = 0$, for all $x = (x^i, x^{-i}) \in C$ and all $i \in I$, then the set ***SEP***(f_i, C_i, I) *admits at least an element.*

The following example shows us that Theorem (⛰) is not a consequence of Theorem (⛅)

Example (Ex. 6)

Consider $C_1 = C_2 = C$ both compact and nonempty subsets of $\mathbb{R}$ and the functions $f_1, f_2 : C^2 \times C \to \mathbb{R}$ defined as

$$f_1(x^1, x^2, y^1) := y^1 - x^2 \quad \text{and} \quad f_2(x^1, x^2, y^2) := y^2 - x^1.$$

Neither f_1 nor f_2 satisfy condition **(♣)**. *However, the bifunction f defined in (⛺) is given by*

$$f_1(x^1, x^2, y^1) + f_2(x^1, x^2, y^2) = \sum_{i=1}^{2} y^i - \sum_{i=1}^{2} x^i.$$

Therefore, the existence of solution of the system of equilibrium problems follows from Theorem ⛰ and not from Theorem ⛅.

Theorem ($\surd$)

For each $i \in I$, *let* C_i *be a nonempty closed subset of a complete metric space* (X_i, d_i), $K_i : C_i \rightrightarrows C_i$ *be a set-valued mapping, and let* $f_i : C \times C_i \to \mathbb{R}$ *be a function such that* (♣) *holds for some transfer lower continuous function* $h_i : C_i \to \mathbb{R}$ *that is also bounded from below. Assume*

1. $\textbf{Fix}(K)$ *is compact;*
2. *for any* $\varepsilon > 0$, *any* $x_0 \in C$, *and any* $i \in I$ *the following implication holds:*

$$\overline{h}_i(x^i) + \varepsilon d_i(x^i, x_0^i) \leq h_i(x_0^i) \implies \exists y^i \in K_i(x^i),\ h_i(y^i) + \varepsilon d_i(x^i, y^i) \leq \overline{h}_i(x^i).$$

Then (♦) *has a* ***solution***.

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Thank you