Can Pourciau's open mapping theorem be derived from Clarke's inverse mapping theorem?

Marián Fabian (joint work with David Bartl)

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Praha - Ballarat

June 30, 2020

Preludium

Statement 1 Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping, defined and C^1-smooth in a neighborhood of the origin, with $f(0) = 0$, and such that the Jacobian $\nabla f(0) \in \mathbb{R}^{n \times n}$ has full rank n.

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Statement 2 Let $m < n$ and let $g\colon \mathbb{R}^n \to \mathbb{R}^m$ be a mapping, defined and C^1-smooth around 0, with $g(0) = 0$, and such $\nabla g(0) \in \mathbb{R}^{m\times n}$ has full rank m.

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How to derive Statement 2 from Statement 1?

First method.

Find a matrix $(n-m)\times n$, say

$$B =: (\, b_{i,j} : m+1 \le i \le n,\ 1 \le j \le n \,) \in \mathbb{R}^{(n-m)\times n},$$

such that $\binom{\nabla g(0)}{B} \in \mathbb{R}^{n\times n}$ has full rank n.

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for all $x \in \mathbb{R}^n$ in the domain of g.

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By Statement 1, there are neighborhoods V and U of the origins in $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$, respectively, and a C^1-smooth mapping $h\colon V \times U \to \mathbb{R}^n$ such that

$$f(h(v,u)) = (v,u) \qquad \text{for every} \quad (v,u) \in V \times U.$$

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Put $\varphi(v) := h(v,0)$ for $v \in V$. Clearly, φ is a C^1-smooth mapping from $V \subset \mathbb{R}^m$ into $\mathbb{R}^n$.

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Moreover, for every $v \in V$ we have $f(\varphi(v)) = f(h(v,0)) = (v,0)$, and so $g(\varphi(v)) = v$. Therefore, φ is the desired right inverse for g. □

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(Yes, if we have the $m \times n$ matrix $\nabla g(0) =: (a_1 \; a_2 \; \ldots \; a_n)$ of full rank m, then there exists $1 \leq k_1 < k_2 < \cdots < k_m \leq n$ such that the square matrix $(a_{k_1} \; a_{k_2} \; \cdots \; a_{k_m})$ has full rank m; this is a deeper fact from linear algebra.)

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Thus f^{-1} exist and $\varphi := i \circ f^{-1}$ is a right inverse to g. □

Enthousiasm

For a mapping $g\colon \mathbb{R}^n \to \mathbb{R}^m$, Lipschitzian in a vicinity of 0, the Clarke generalized Jacobian $\partial g(0)$ of g at 0 is defined as the (closed) convex hull of all possible limits $\lim_{k\to\infty} \nabla g(x_k)$, where we take only those $x_k \in \mathbb{R}^n$ where the derivative $\nabla g(x_k)$ exists.

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Theorem 1 (Clarke [C, Theorem 7.1.1], 1976)

Let $f \colon \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitzian mapping defined in a neighborhood of 0*, with $f(0) = 0$, and such that every matrix from $\partial f(0)$ has rank n.*

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Theorem 2 (Pourciau [P], 1977)

Consider $m, n \in \mathbb{N}$ such that $m < n$ and let $g\colon \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitzian mapping defined in a neighborhood of 0, with $g(0) = 0$, and such that every matrix from $\partial g(0)$ has full rank m.

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Then g has near 0 a right inverse, that is, there are a neighbourhood V of 0 in $\mathbb{R}^m$ and a mapping $\varphi\colon V \to \mathbb{R}^n$ such that $g(\varphi(v)) = v$ for every $v \in V$.

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Lemma 3

Consider $m, n \in \mathbb{N}$ such that $m < n$ and let $\mathcal{A}$ be a convex compact set in $\mathbb{R}^{m\times n}$ consisting of $m \times n$ matrices, each having full rank m. Then

(i) *there exists a matrix $B \in \mathbb{R}^{(n-m)\times n}$ of full rank $n - m$ such that for every $A \in \mathcal{A}$ the augmented square matrix $\binom{A}{B}$ has full rank n, or*

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(ii) *there exists a linear subspace* $0 \in W \subset \mathbb{R}^{n\times 1}$, *of dimension* m, *such that for every* $A \in \mathcal{A}$ *the mapping* $A_{|W} : W \longrightarrow \mathbb{R}^{m\times 1}$ *is surjective.*

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Proof of Theorem 2 by using Theorem 1 and Lemma 3 (i).

Assume that g has the form $g(x) =: (g_1(x), \ldots, g_m(x))$ whenever $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$ belongs to the domain of g. Let $B =: (b_{i,j} : m + 1 \leq i \leq n,\ 1 \leq j \leq n)$ be the matrix found for the (convex compact) set $\mathcal{A} := \partial g(0)$ by Lemma 3.

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Now, Theorem 1 provides neighborhoods V and U of the origins in $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$, respectively, and a continuous mapping $h\colon V \times U \to \mathbb{R}^n$ such that $f(h(v,u)) = (v,u)$ for every $(v,u) \in V \times U$. Put $\varphi(v) := h(v, 0)$, $v \in V$. □

Proof of Theorem 2 by using Theorem 1 and Lemma 3 (ii).

Similar to the second method of derivation of Statement 2 from Statement 1.

□

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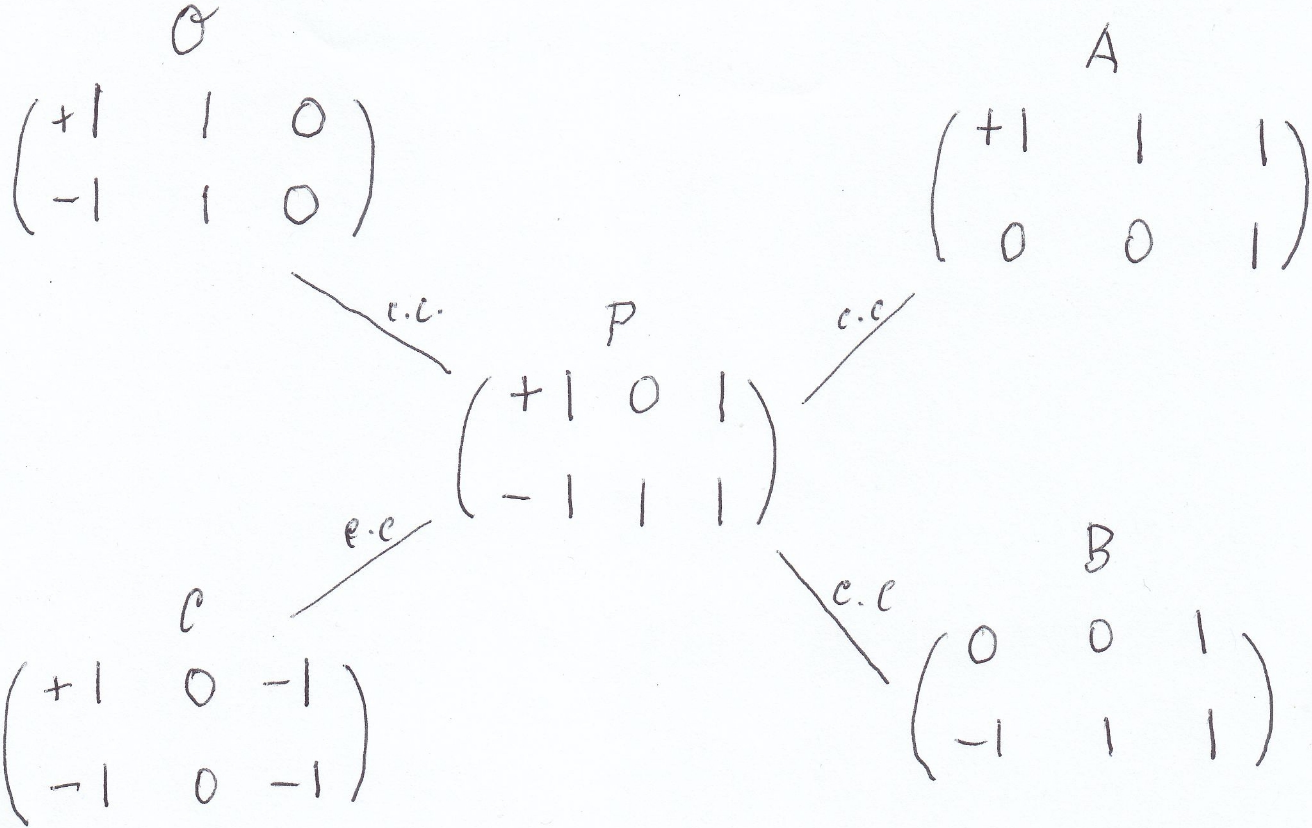
O
+1 1 0
-1 1 0
A
+1 1 1
0 0 1
c.c.
c.c
P
+1 0 1
-1 1 1
e.c
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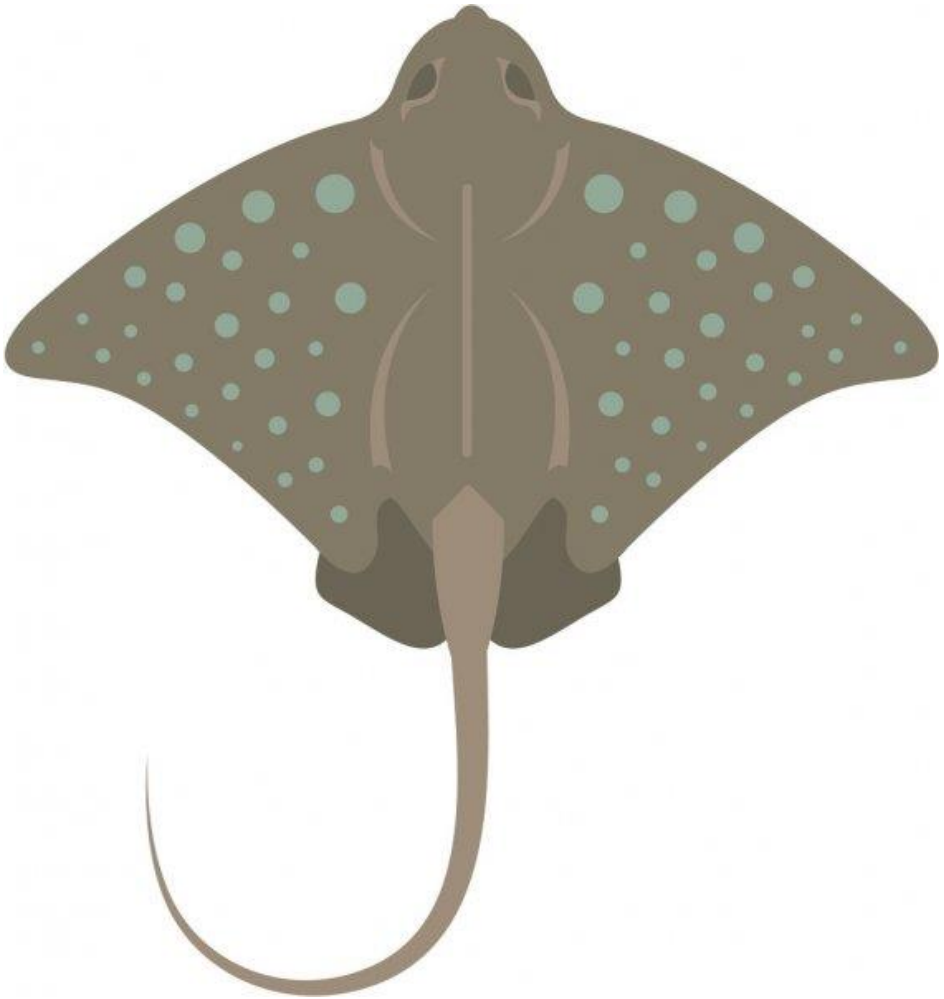
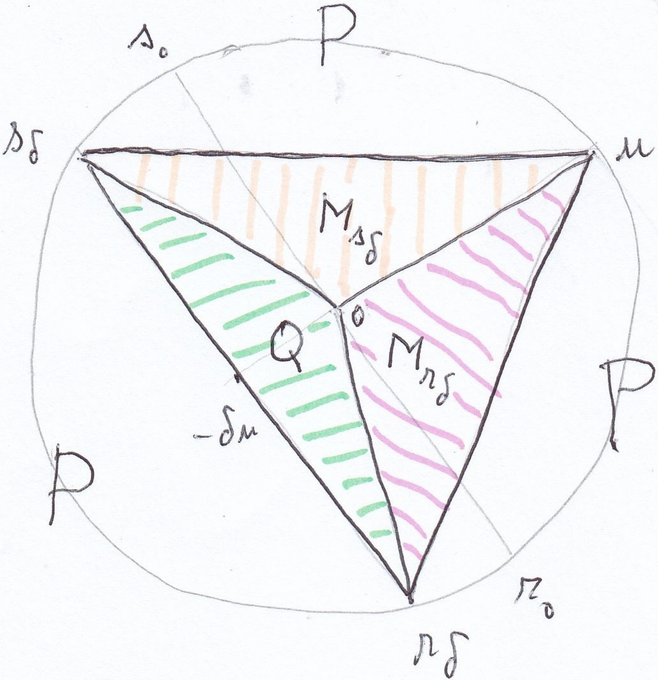


$0 < \delta < 1$

2D RAY-FISH

$P, Q, M_{r\delta}, M_{s\delta} \in \mathbb{R}^{2\times 2}$



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this is a well defined, piecewise linear, mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$*, with a Lipschitzian constant* $\max\{\|P\|, \|Q\|, \|M_{r_\delta}\|, \|M_{s_\delta}\|\} =: L_\delta$*,*

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$$f(x) := \begin{cases} P(x) & \text{if } x \in \mathbb{R}^2 \setminus \mathrm{co}\,\{r_\delta, s_\delta, u\}, \ \left(\subset \mathbb{R}^2 \setminus B_{\mathbb{R}^2}\right) \\ Q(x) - \delta(P-Q)(u) & \text{if } x \in \mathrm{co}\,\{r_\delta, s_\delta, 0\}, \\ M_{r_\delta}(x) - \delta(P-Q)(u) & \text{if } x \in \mathrm{co}\,\{r_\delta, u, 0\}, \\ M_{s_\delta}(x) - \delta(P-Q)(u) & \text{if } x \in \mathrm{co}\,\{s_\delta, u, 0\}, \end{cases}$$

this is a well defined, piecewise linear, mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$, with a Lipschitzian constant $\max\{\|P\|, \|Q\|, \|M_{r_\delta}\|, \|M_{s_\delta}\|\} =: L_\delta$, and such that

$$\left\{ \nabla f(x) : \ f \text{ is differentiable at } x \in \mathbb{R}^2 \right\} = \{P, Q, M_{r_\delta}, M_{s_\delta}\}.$$

Lemma 6 (2D Ray-fish)

Let $P, Q \in \mathbb{R}^{2\times 2}$ be two matrices with a continuous contact. Pick an r_0 in the doubleton $(P-Q)^{-1}(0) \cap S_{\mathbb{R}^2}$, put $s_0 := -r_0$, and pick a u in the doubleton $((P-Q)^{-1}(0))^{\perp} \cap S_{\mathbb{R}^2}$. Consider any $\delta \in (0,1)$ and let r_δ and s_δ be the two elements of the doubleton $((P-Q)^{-1}(0) - \delta u) \cap S_{\mathbb{R}^2}$ such that $r_\delta \in \widehat{-u, r_0}$ and $s_\delta \in \widehat{-u, s_0}$.
Then there exist unique matrices $M_{r_\delta}, M_{s_\delta} \in \mathbb{R}^{2\times 2}$ such that

$$M_{r_\delta}(r_\delta) = Q(r_\delta), \quad M_{s_\delta}(s_\delta) = Q(s_\delta), \quad M_{r_\delta}(u) = M_{s_\delta}(u) = P(u) + \delta(P-Q)(u).$$

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this is a well defined, piecewise linear, mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$, with a Lipschitzian constant $\max\{\|P\|, \|Q\|, \|M_{r_\delta}\|, \|M_{s_\delta}\|\} =: L_\delta$, and such that

$$\left\{ \nabla f(x) : \ f \text{ is differentiable at } x \in \mathbb{R}^2 \right\} = \{P, Q, M_{r_\delta}, M_{s_\delta}\}.$$

Finally, for $\delta \downarrow 0$ we have $M_{r_\delta} \longrightarrow P$, $M_{s_\delta} \longrightarrow P$, and $L_\delta \longrightarrow \max\{\|P\|, \|Q\|\}$.

PROOF

$$M_{r_\delta}(r_\delta \ \mu) = (Q r_\delta \ \ P\mu + \delta(P-Q)\mu)$$

$$M_{r_\delta} = (Q r_\delta \ \ P\mu + \delta(P-Q)\mu)(r_\delta \ \mu)^{-1}$$

$$\Big\downarrow \text{ as } \delta \downarrow 0$$

$$(Q r_0 \ \ P\mu)(r_0 \ \mu)^{-1}$$

$$\|$$

$$(P r_0 \ \ P\mu)(r_0 \ \mu)^{-1} = P$$

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It looks as follows

Lemma 7 (3D Ray-fish)

Let $P, Q \in \mathbb{R}^{2\times 3}$ *be two matrices with a continuous contact. Let* u *be an element of the doubleton* $((P-Q)^{-1}(0))^{\perp} \cap S_{\mathbb{R}^3}$ *and pick three points* r_0, s_0, t_0 *in the circle* $((P-Q)^{-1}(0)) \cap S_{\mathbb{R}^3}$ *such that* $\operatorname{co}\{r_0, s_0, t_0\}$ *forms an equilateral triangle. Consider any* $\delta \in (0,1)$ *and let* $r_\delta \in \widehat{-u, r_0}$, $s_\delta \in \widehat{-u, s_0}$, $t_\delta \in \widehat{-u, t_0}$ *be the unique points lying in the circle* $((P-Q)^{-1}(0) - \delta u) \cap S_{\mathbb{R}^3}$.

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Moreover, putting

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Simplified 3D Rey fish Lemma 7: Given $P, Q \in \mathbb{R}^{2\times 3}$ and $\delta \in (0,1)$, there are matrices $M_{r_\delta s_\delta}, M_{s_\delta t_\delta}, M_{t_\delta r_\delta} \in \mathbb{R}^{2\times 3}$ such that they converge to the matrix P and . . . .

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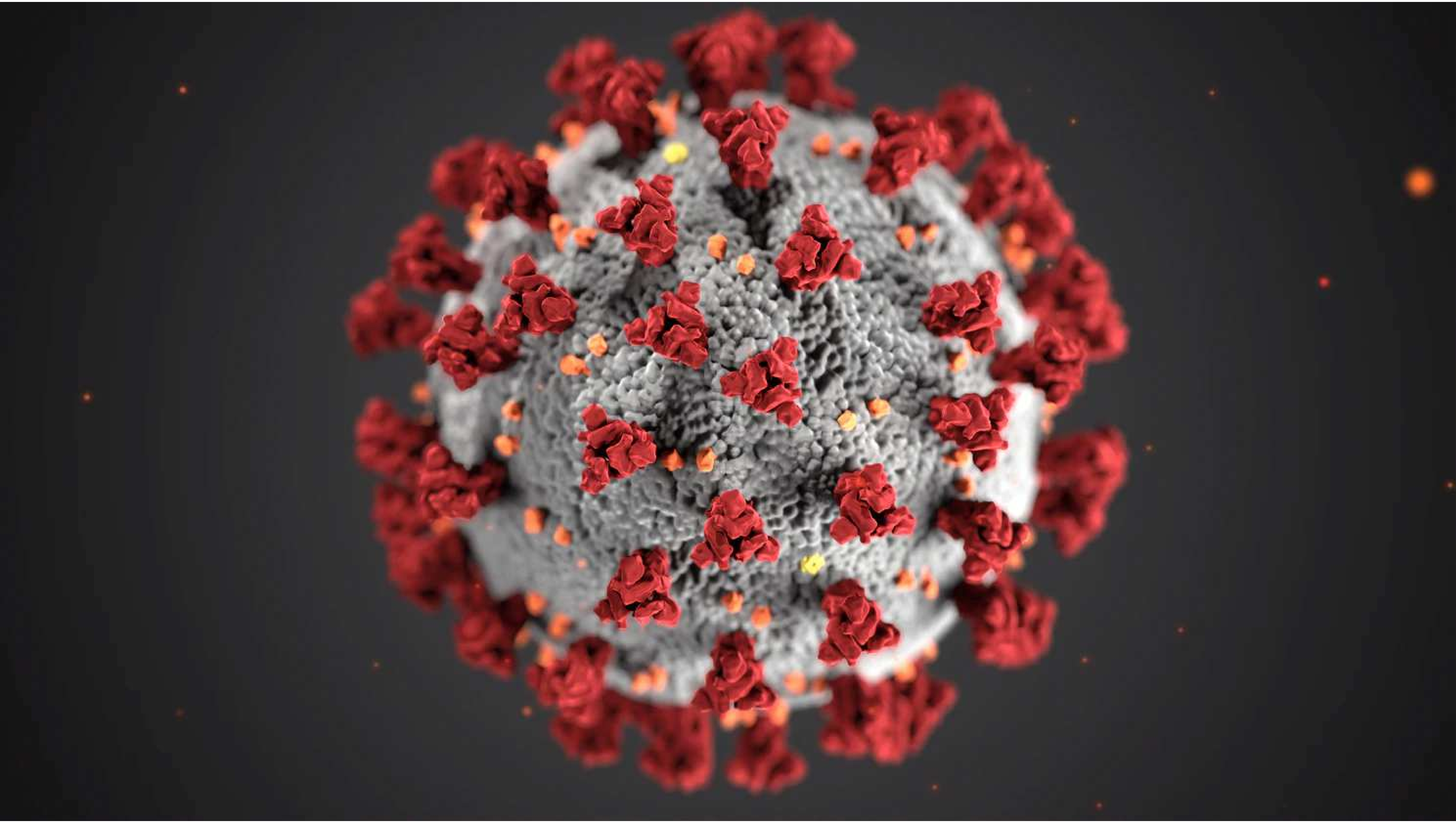
and that, for every $0 \ne x \in \mathbb{R}^3$ *there exists an* $\alpha > 0$ *such that* $\alpha x \in T(\beta,\gamma)$*, the mapping* h *is differentiable at* αx *and* $\nabla h(\alpha x) = Q$.



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Proof.

Consider countably many diminishing coronas converging to the origin. □

Conclusion

By Pourciau's Theorem 2, the Lipschitzian mapping $g\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, with $g(0) = 0$, provided by Theorem 9, admits a right inverse in the vicinity of 0.

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In particular, for $W := \{(x_1, x_2, 0) :\ x_1, x_2 \in \mathbb{R}\}$, we find $M \in \partial g(0)$ such that $(0, 0, 1) \in \operatorname{lin}\{m_1, m_2\}$; then $L(w) := Mw,\ w \in W$, "works".

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- B.H. Pourciau, Analysis and optimization of Lipschitz continuous mappings, *J. Optimization Theory Appl.* 22 (1977), no. 3, 311–351.

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Thank you for your attention