Zero Duality Gap Conditions via Abstract Convexity

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Convex functions

Function $f : X \to \mathbb{R}_{+\infty}$ is l.s.c convex if and only if

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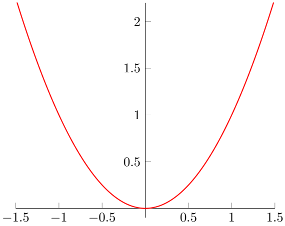

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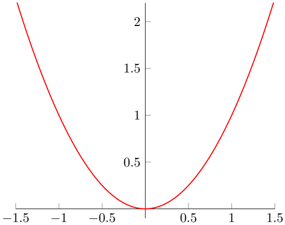

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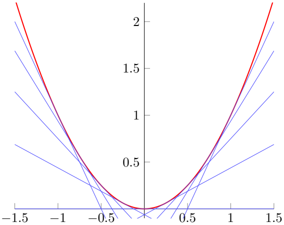

Figure 2: Support affine functions of x^2, $\{2ax - a^2 : a \in \mathbb{R}\}$.

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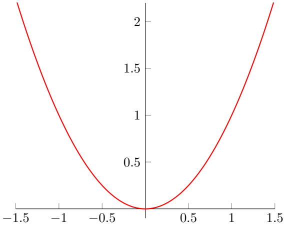

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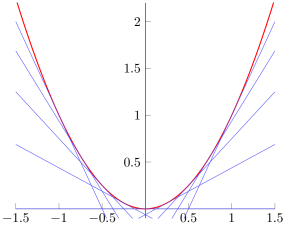

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The space of affine functions

$$H := \{at + b : a, b \in \mathbb{R}\}.$$

Nonconvex functions

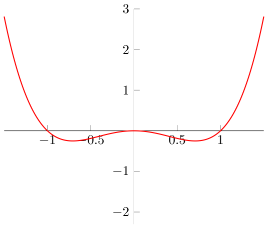

Figure 3: $x^4 + x^2$.

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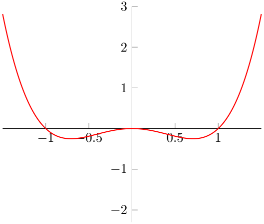

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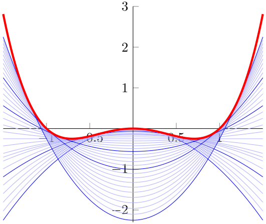

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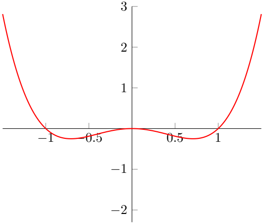

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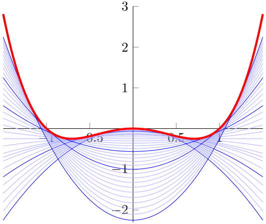

Figure 4: Quadratic functions support $x^4 + x^2$.

$$H := \{a\textcolor{red}{t} + b : a, b \in \mathbb{R}\};$$
$$\mathcal{H} := \{a\textcolor{red}{t^2} + b : a, b \in \mathbb{R}\}.$$

Outline

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Abstract linear functions

$X-$nonempty, $\mathcal{F} := X^{\mathbb{R}}-$the set of all functions from X to $\mathbb{R}$.

Definition

A *space of abstract linear functions*, denoted by $\mathcal{L}$, is a subset of $\mathcal{F}$ that satisfies the following properties.

(a) $\mathcal{L}$ is closed w.r.t the addition operator i.e. $f_1, f_2 \in \mathcal{L} \Longrightarrow f_1 + f_2 \in \mathcal{L}$.

(b) For every $l \in \mathcal{L}$ and $m \in \mathbb{N}$, there exist $l_1, \ldots, l_m \in \mathcal{L}$ s.t.

$$l = l_1 + \ldots + l_m.$$

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If $0 \in \mathcal{L}$ and $\mathcal{L}$ verifies (a), then $\mathcal{L}$ verifies (b).

Abstract affine functions

$X-$nonempty, $\mathcal{F} := X^{\mathbb{R}}$, $\mathcal{L}-$space of abstract linear functions, $f : X \to \mathbb{R}$, $C \subset \mathcal{L}$.

Definition

The space of abstract affine functions is defined as $\mathcal{H} := \{l + c : l \in \mathcal{L},\ c \in \mathbb{R}\}$.

1. The set $\operatorname{supp} f := \{h \in \mathcal{H} : h(x) \le f(x), \forall x \in X\}$ is the support set of f.
2. f is $\mathcal{H}-$convex if there is a subset $H \subset \mathcal{H}$ such that

$$f(x) = \sup_{h \in H} h(x), \quad \forall x \in X.$$

Subdifferentials, infimal convolution and Fenchel conjugate

$X-$nonempty, $\mathcal{F} := X^{\mathbb{R}}$, $\mathcal{L}-$space of abstract linear functions, $f : X \to \mathbb{R}$, $\varepsilon \geq 0$.

Definition

1. Given m functions $\psi_1, \ldots, \psi_m : X \to \mathbb{R}_{+\infty}$, the infimal convolution of the functions $\psi_1, \ldots, \psi_m$ is the function $\psi_1 \Box \ldots \Box \psi_m : X \to \mathbb{R}_{\pm\infty}$ defined by

$$\psi_1 \Box \ldots \Box \psi_m(x) := \inf_{x_1 + \ldots + x_m = x} \{\psi_1(x_1) + \ldots + \psi_m(x_m)\}.$$

2. The Fenchel conjugate of f is the function $f^* : \mathcal{L} \to \mathbb{R}_{\pm\infty}$, defined as

$$f^*(l) := \sup_{x \in X} \{l(x) - f(x)\}.$$

3. The $\varepsilon-$subdifferential of the function f at a point $x \in \operatorname{dom} f$ is the mapping $\partial_\varepsilon f : X \rightrightarrows \mathcal{L}$ defined as

$$\partial_\varepsilon f(x) := \{l \in \mathcal{L} : f(y) - f(x) - (l(y) - l(x)) + \varepsilon \geq 0 \text{ for all } y \in X\}.$$

Basic properties

1. f is $\mathcal{H}-$convex, f attains global minimum at $\bar{x}$ iff $0 \in \partial f(\bar{x})$.

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4. $$\left(\sum_{i=1}^{m} f_i\right)^{*} \le f_1^{*}\Box \dots \Box f_m^{*}.$$
5. Sum rule
   $$\bigcap_{\eta>0} \bigcup_{\substack{\varepsilon_i \ge 0,\ i=1,\dots,n \\ \sum_{i=1}^{m} \varepsilon_i = \varepsilon+\eta}} \sum_{i=1}^{m} \partial_{\varepsilon_i} f_i(x) \subset \partial_{\varepsilon}\left(\sum_{i=1}^{m} f_i\right)(x).$$

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Abstract convex set

Definition

A set $C \subset \mathcal{L}$ is $\mathcal{L}-$convex if for any $l_0 \notin C$, there is an $x \in X$ such that

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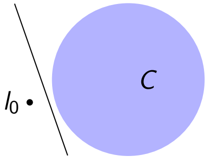

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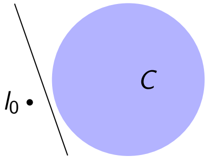

A set $C \subset \mathcal{L}$ is $\mathcal{L}-$convex iff there is an $\mathcal{L}-$convex function $f : X \to \mathbb{R}_{+\infty}$ such that $C = \mathrm{supp}_{\mathcal{L}} f$.

Example

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$X := \mathbb{R}$, $\mathcal{L} := \{\phi_a : \phi_a(t) := at^2, a \in \mathbb{R}\}$ is a space of abstract linear functions.

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$X := \mathbb{R}$, $\mathcal{L} := \{\phi_a : \phi_a(t) := at^2, a \in \mathbb{R}\}$ is a space of abstract linear functions. Then, $\mathcal{H} := \{\Psi_{a,b} : \Psi_{a,b}(t) := at^2 + b, a, b \in \mathbb{R}\}$ is the space of abstract affine functions.

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$C := \{\Psi_{a,b} : b \leq \frac{1}{a},\ a < 0\}$ is the support set of function $f(x) = -2|x|$.

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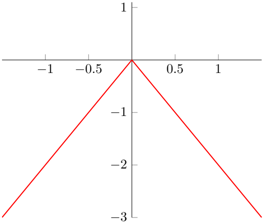

Figure 5: $-2\,|x|$.

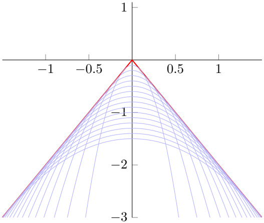

Figure 6: Support set of $-2\,|x|$.

Subdifferentials and Fenchel conjugate

The subdifferential and Fenchel conjugate of f

$$\partial f(x) = \begin{cases} \emptyset & x = 0, \\ \left\{ \phi_a : a = -\dfrac{1}{|x|} \right\} & \text{otherwise}; \end{cases}$$

$$f^*(\phi_a) = \sup_{x \in \mathbb{R}} \{\phi_a(x) - f(x)\} = \begin{cases} +\infty & a \geq 0, \\ -\dfrac{1}{a} & a < 0; \end{cases}$$

where $\phi_a(t) = at^2$, $a \in \mathbb{R}$.

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More example

Example

$X = \mathbb{R}^n$, $\mathcal{L} := \{a_1 x_1^2 + \ldots + a_n x_n^2 : a_1, \ldots, a_n \in \mathbb{R}\}$,
$\mathcal{H} = \{a_1 x_1^2 + \ldots + a_n x_n^2 + b : a_1, \ldots, a_n,\ b \in \mathbb{R}\}$.

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Example

$X = \mathbb{R}^n$, $\mathcal{L} = \left\{ l := \sum_{i=0}^{n} a_i h_i \right\}$ where $a_i \in \mathbb{R}$, $i = 1, \ldots, n$ and

$$h_0(x) = \|x\|^2, \quad h_1(x) = x_1, \ldots, h_n(x) = x_n.$$

Then, $\mathcal{H} := \{h : h = \sum_{i=0}^{n} a_i h_i + c, c \in \mathbb{R}\}$. If a l.s.c function f is bounded from below by a function in $\mathcal{H}$, then the function f is $\mathcal{H}-$convex.

Maintenance

Calculus Rules

$$\partial_{\varepsilon}(f+g)(x) \supset \bigcap_{\eta>0} \bigcup_{\substack{\varepsilon_1,\varepsilon_2\geq 0 \\ \varepsilon_1+\varepsilon_2=\varepsilon+\eta}} \left(\partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x)\right).$$

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Theorem (Jeyakumar et al, 2007)

Let X be a set and let $\mathcal{L}$ be a set of abstract linear functions on X, $f, g : X \to \mathbb{R}_{+\infty}$ be $\mathcal{H}-$convex functions, $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset$. Then, equality

$$\mathrm{epi}\, f^* + \mathrm{epi}\, g^* = \mathrm{epi}\, (f+g)^*$$

holds iff for any $\varepsilon \geq 0$,

$$\partial_\varepsilon (f+g)(x) = \bigcup_{\varepsilon_1+\varepsilon_2=\varepsilon, \varepsilon_1,\varepsilon_2\geq 0} \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x), \quad x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g.$$

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$\operatorname{epi} f^* + \operatorname{epi} g^* = \operatorname{epi} (f+g)^* \iff (f+g)^* = f^* \square g^*$ with exact infimal.

Pointwise convergence topology

$$\partial_{\varepsilon}(f+g)(x) = \bigcap_{\eta>0} \operatorname{cl}^{*}\left(\bigcup_{\substack{\varepsilon_1,\varepsilon_2=\varepsilon+\eta \\ \varepsilon_1,\varepsilon_2\geq 0}} \left(\partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x)\right) \right).$$

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On $\mathcal{F}$, consider the weakest topology that makes all the functions $x : \mathcal{F} \to \mathbb{R}, f \mapsto f(x)$ continuous.

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On $\mathcal{F}$, consider the weakest topology that makes all the functions $x : \mathcal{F} \to \mathbb{R}, f \mapsto f(x)$ continuous.

The set $\mathcal{L}$ satisfies the following conditions:

A- $\mathcal{L}$ is an $\mathbb{R}-$vector space;

B- $\mathcal{L}$ is closed in $\mathcal{F}$ w.r.s pointwise convergence topology.

Then $\mathcal{L}$ is locally convex space.

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6. for any functions f, g, $x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$, and $\varepsilon \geq 0$,

$$\bigcap_{\eta>0} \mathrm{cl} \bigcup_{\substack{\varepsilon_1,\varepsilon_2 \geq 0 \\ \varepsilon_1+\varepsilon_2=\varepsilon+\eta}} \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x) \subset \partial_\varepsilon (f+g)(x);$$

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7. Let $F, G \in \mathcal{F}$ and $G \leq F$. Then the set

$$A := \{f \in \mathcal{F} : G \leq f \leq F\}$$

is compact.

Sum rule

$X-$nonempty, $\mathcal{F} := X^{\mathbb{R}}$, $\mathcal{L}-$space of abstract linear functions, weak* closed in $\mathcal{F}$.

Theorem

Given two functions $f, g : X \to \mathbb{R}_\infty$, $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$, *assume that*

$$\operatorname{cl}\left(\operatorname{epi} f^* + \operatorname{epi} g^*\right) = \operatorname{epi}\left(f+g\right)^*.$$

Then, for any number $\varepsilon \geq 0$, *for all* $x \in \operatorname{dom} f \cap \operatorname{dom} g$,

$$\partial_\varepsilon\left(f+g\right)(x) = \bigcap_{\eta>0} \operatorname{cl}\left(\bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon+\eta \\ \varepsilon_1,\varepsilon_2\geq 0}} \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x)\right)$$

holds for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$.

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Zero Duality Gap

Given m functions $f, \ldots, f_m : X \to \mathbb{R}_{+\infty}$ $(m \geq 2)$, consider the problem

$$\inf \left(\sum_{i=1}^{m} f_i(x) \right), \tag{P}$$
$$\text{s.t.} \quad x \in X.$$

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$$\text{s.t.} \quad x \in X.$$

The dual problem of (P) is given as follows:

$$\sup \left(\sum_{i=1}^{m} -f_i^*(l_i) \right), \tag{D}$$
$$\text{s.t.} \quad l_1, \ldots, l_m \in \mathcal{L},$$
$$l_1 + \ldots + l_m = 0.$$

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$$\text{s.t.} \quad x \in X.$$

The dual problem of (P) is given as follows:

$$\sup \left(\sum_{i=1}^{m} -f_i^*(l_i) \right), \tag{D}$$
$$\text{s.t.} \quad l_1, \ldots, l_m \in \mathcal{L},$$
$$l_1 + \ldots + l_m = 0.$$

Denote by $v(P)$, $v(D)$, the optimal values of (P) and (D).

Zero duality gap holds if $v(P) = v(D)$. In general, $v(P) \geq v(D)$.

Zero Duality Gap and Infimal Convolution

$$v(P) = \inf_{x \in X} \left(\sum_{i=1}^{m} f_i(x) \right) = - \left(\sum_{i=1}^{m} f_i \right)^* (0); \tag{P1}$$

$$v(D) = \sup_{l_1 + \ldots + l_m = 0} \left(\sum_{i=1}^{m} -f_i^*(l_i) \right) = -(f_1^* \Box \ldots \Box f_m^*)(0). \tag{D1}$$

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Recall: $-\left(\sum_{i=1}^{m} f_i \right)^* \geq -f_1^* \square \ldots \square f_m^*.$

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Borwein, Burachik and Yao (2014)

$$\left(\sum_{i=1}^{m} f_i \right)^* = f_1^* \Box \ldots \Box f_m^*.$$

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Conditions for Zero Duality Gap I

$X-$nonempty, $\mathcal{F} := X^{\mathbb{R}}$, $\mathcal{L}-$space of abstract linear functions.

Let $f_i : X \to \mathbb{R}_{+\infty}$ where $(i = 1, \dots, m)$ $(m \geq 2)$ with $\bigcap_{i=1}^{m} \mathrm{dom}\, f_i \neq \emptyset$. Consider the following five conditions:

1. $\left(\sum_{i=1}^{m} f_i\right)^* = f_1^* \square \dots \square f_m^*$ in $\mathcal{L}$;
2. For every $x \in X$ and $\varepsilon \geq 0$,

$$\partial_\varepsilon (f_1 + \dots + f_m)(x) = \bigcap_{\eta > 0} \left[\bigcup_{\substack{\varepsilon_1 + \dots + \varepsilon_m = \varepsilon + \eta \\ \varepsilon_i \geq 0}} (\partial_{\varepsilon_1} f_1(x) + \dots + \partial_{\varepsilon_m} f_m(x)) \right].$$

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Conditions for Zero Duality Gap II

3. There exists $K > 0$ such that for every $x \in \bigcap_{i=1}^{m} \operatorname{dom} f_i$, and every $\varepsilon > 0$

$$\partial_{\varepsilon}\left(\sum_{i=1}^{m} f_i\right)(x) \subset \sum_{i=1}^{m} \partial_{K\varepsilon} f_i(x).$$

4. There exists $K > 0$ such that for every $x \in \bigcap_{i=1}^{m} \operatorname{dom} f_i$, and every $\varepsilon > 0$

$$\operatorname{cl}\left[\sum_{i=1}^{m} \partial_{\varepsilon} f_i(x)\right] \subset \sum_{i=1}^{m} \partial_{K\varepsilon} f_i(x).$$

5. $f_1^* \square \ldots \square f_m^*$ is lower semicontinuous;

We have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v).

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Conditions for Zero Duality Gap

If the sum rule

$$\partial_{\varepsilon}(f+g)(x) = \bigcap_{\eta>0} \operatorname{cl}\left(\bigcup_{\substack{\varepsilon_1+\varepsilon_2=\varepsilon+\eta \\ \varepsilon_1,\varepsilon_2\geq 0}} \partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x) \right)$$

holds, then all five statements are equivalent.

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Sufficient and necessary conditions

$X-$nonempty, $\mathcal{F} := X^{\mathbb{R}}$, $\mathcal{L}-$space of abstract linear functions. Let $f_i : X \to \mathbb{R}_{+\infty}$ $(i = 1, \ldots, m)$ $(m \geq 2)$ with $\bigcap_{i=1}^{m} \mathrm{dom}\, f_i \neq \emptyset$.

Theorem

The following conditions are equivalent.

1. *For all $\varepsilon > 0$, there exists an $x \in X$ such that*
$$\partial_\varepsilon f_1(x) + \ldots + \partial_\varepsilon f_m(x) \ni 0.$$
2. $$\left(\sum_{i=1}^{m} f_i\right)^* (0) = f_1^* \square \ldots \square f_m^*(0) < +\infty.$$

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$X-$nonempty, $\mathcal{F} := X^{\mathbb{R}}$, $\mathcal{L}-$space of abstract linear functions. Let $f_i : X \to \mathbb{R}_{+\infty}$ $(i = 1, \ldots, m)$ $(m \geq 2)$ with $\bigcap_{i=1}^{m} \operatorname{dom} f_i \neq \emptyset$.

Theorem

The following conditions are equivalent. Furthermore, the x for which condition (*) holds is a solution of problem (P).

1. *There exists an $x \in X$ such that*
$$\partial_\varepsilon f_1(x) + \ldots + \partial_\varepsilon f_m(x) \ni 0. \tag{*}$$
2. $\left(\sum_{i=1}^{m} f_i\right)^*(0) = f_1^* \square \ldots \square f_m^*(0) < +\infty.$

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Example

Example

$X := \mathbb{R}$, $\mathcal{L} := \{\phi_a : \phi_a(t) := at^2, a \in \mathbb{R}\}$ is a space of abstract linear functions.
$\mathcal{H} := \{\Psi_{a,b} : \Psi_{a,b}(t) := at^2 + b, a, b \in \mathbb{R}\}$ is the space of abstract affine functions.

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Let f, g be any $\mathcal{H}-$convex functions, we have $\mathrm{cl}\,(\mathrm{supp}\, f + \mathrm{supp}\, g)$ is $\mathcal{H}-$convex. Consequently,

$$\mathrm{cl}\,(\mathrm{epi}\, f^* + \mathrm{epi}\, g^*) = \mathrm{epi}\,(f+g)^*.$$

The sum rule holds for every $\mathcal{H}-$convex functions.

Example

Consider three $\mathcal{H}-$convex functions

$$f_1(x) := x^4 - x^2;$$

$$f_2(x) := 1 - 2|x|;$$

$$f_3(x) := \begin{cases} 1 - 2|x| & -0.5 \le x \le 0.5, \\ 0 & \text{otherwise.} \end{cases}$$

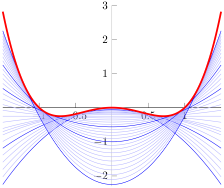

Figure 7: Support set of f_1.

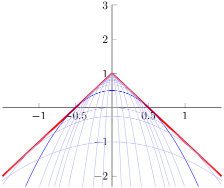

Figure 8: Support set of f_2.

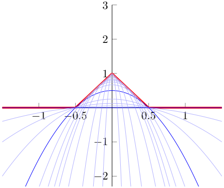

Figure 9: Support set of f_3.

Examples

Their conjugate functions

$$f_1^*(\phi_a) = \sup_{x\in\mathbb{R}} \{\phi_a(x) - f_1(x)\} = \begin{cases} \dfrac{(a+1)^2}{4} & a \geq -1, \\ 0 & a < -1; \end{cases}$$

$$f_2^*(\phi_a) = \sup_{x\in\mathbb{R}} \{\phi_a(x) - f_2(x)\} = \begin{cases} +\infty & a \geq 0, \\ -1 - \dfrac{1}{a} & a < 0; \end{cases}$$

$$f_3^*(\phi_a) = \sup_{x\in\mathbb{R}} \{\phi_a(x) - f_3(x)\} = \begin{cases} +\infty & a > 0, \\ \dfrac{a}{4} & a \in [-2, 0], \\ -1 - \dfrac{1}{a} & a < -2. \end{cases}$$

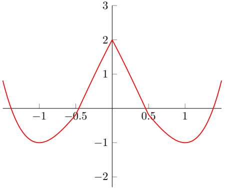

Figure 10: Sum function $f_1 + f_2 + f_3$.

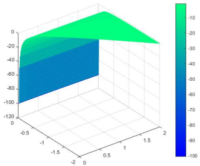

Figure 11: Dual function $-f_1^* - f_2^* - f_3^*$ in the subspace $\phi_{a_1} + \phi_{a_2} + \phi_{a_3} = 0$.

The minimization problem

$$v(p) := \min \left(f_1 + f_2 + f_3 \right), \tag{p}$$

has the dual problem

$$v(d) := \sup_{\phi_{a_1}+\phi_{a_2}+\phi_{a_3}=0} \left(-f_1^*(\phi_{a_1}) - f_2^*(\phi_{a_2}) - f_3^*(\phi_{a_3}) \right). \tag{d}$$

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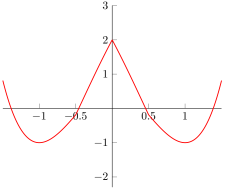

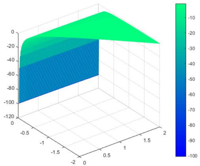

Figure 10: Sum function $f_1 + f_2 + f_3$.

Figure 11: Dual function $-f_1^* - f_2^* - f_3^*$ in the subspace $\phi_{a_1} + \phi_{a_2} + \phi_{a_3} = 0$.

The minimization problem

$$v(p) := \min\left(f_1 + f_2 + f_3\right), \qquad \text{(p)}$$

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$$v(d) := \sup_{\phi_{a_1}+\phi_{a_2}+\phi_{a_3}=0} \left(-f_1^*(\phi_{a_1}) - f_2^*(\phi_{a_2}) - f_3^*(\phi_{a_3})\right). \qquad \text{(d)}$$

Apply condition (*) for $x = \pm 1$, we have the zero duality gap holds, moreover $x = \pm 1$ are the solutions of (p).

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Thank You!