A general branch-and-bound framework for global multiobjective optimization

— a picture book —

Oliver Stein

Institute of Operations Research
Karlsruhe Institute of Technology (KIT)

Variational Analysis and Optimisation Webinar
July 22, 2020

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May contain traces of Variational Analysis

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This is joint work with

Prof. Dr. Gabriele Eichfelder,

Dr. Peter Kirst

and

M.Sc. Laura Meng.

(Optimization Online, Preprint ID 2020-07-7909)

Agenda

Framework

We consider multiobjective optimization problems of the form

$$MOP: \qquad \min\ f(x) \quad \text{s.t.} \quad g(x) \leq 0,\ x \in X$$

with

- $f: \mathbb{R}^n \to \mathbb{R}^m$ continuous,
- $g: \mathbb{R}^n \to \mathbb{R}^k$ continuous,
- $X = [\underline{x}, \overline{x}]$ an n-dimensional box with $\underline{x}, \overline{x} \in \mathbb{R}^n$, $\underline{x} \leq \overline{x}$,
- $M = \{x \in X \mid g(x) \leq 0\}$.

No convexity assumptions.

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No convexity assumptions.

A missing piece in multiobjective B&B

The branch-and-bound idea from the global solution of single objective optimization problems has been adapted to *MOP* by various authors (see next slide).

In particular, partial lower bounds and overall upper bounds were introduced to design discarding tests.

However, general overall lower bounds have not been obtained from partial lower bounds, and the resulting enclosures of the nondominated set have not been employed for node selection and termination criteria.

In this talk we propose a way to close this gap.

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Literature

Yu. G. Evtushenko, M. A. Posypkin, *Method of non-uniform coverages to solve the multicriteria optimization problems with guaranteed accuracy*, Automation and Remote Control, Vol. 75 (2014), 1025–1040.

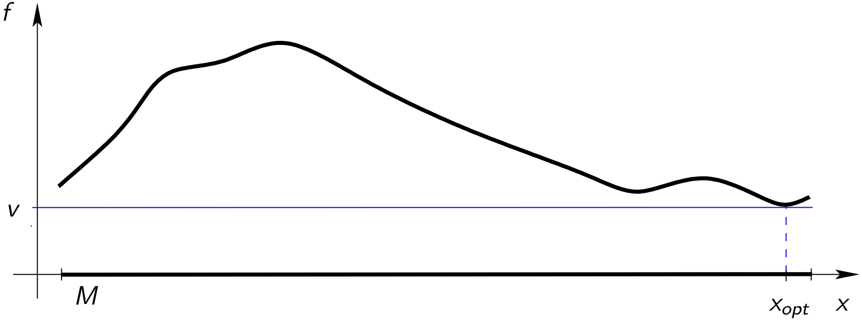
J. Fernández, B. Tóth, *Obtaining the efficient set of nonlinear biobjective optimization problems via interval branch-and-bound methods*, Comput. Optim. Appl., Vol. 42 (2009), 393–419.

J. Niebling, G. Eichfelder, *A branch-and-bound based algorithm for nonconvex multiobjective optimization*, SIAM J. Optim., Vol. 29 (2019), 794–821.

D. Scholz, *The multicriteria big cube small cube method*, TOP, Vol. 18 (2010), 286–302.

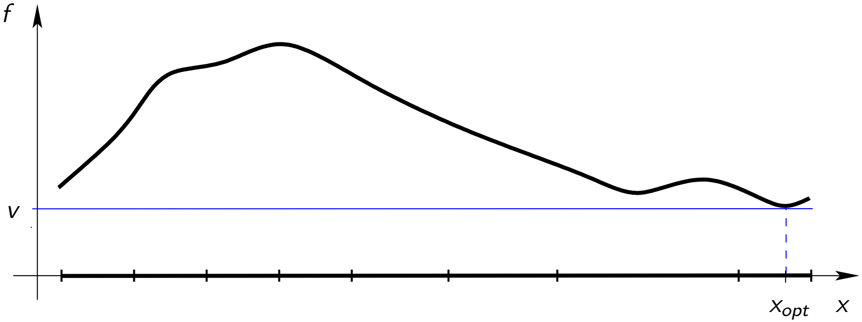
A. Žilinskas, J. Žilinskas, *Adaptation of a one-step worst-case optimal univariate algorithm of bi-objective Lipschitz optimization to multidimensional problems,* Commun. Nonlinear Sci. Numer. Simulat., Vol. 21 (2015), 89–98.

Single objective B&B – optimality



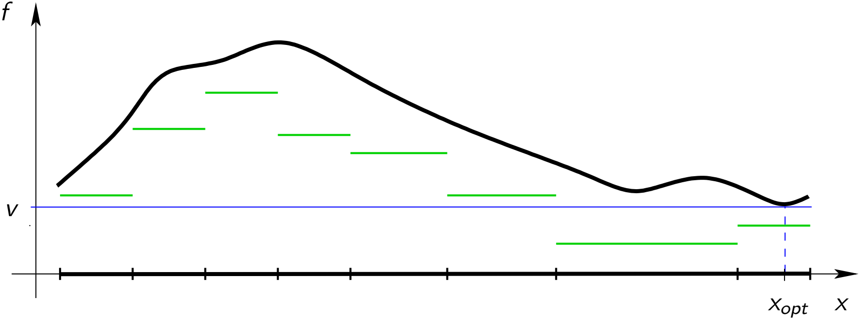
x_{opt} is optimal since no $x \in M$ satisfies $f(x) < f(x_{opt}) =: v$.

Single objective B&B – subdivision of the feasible set



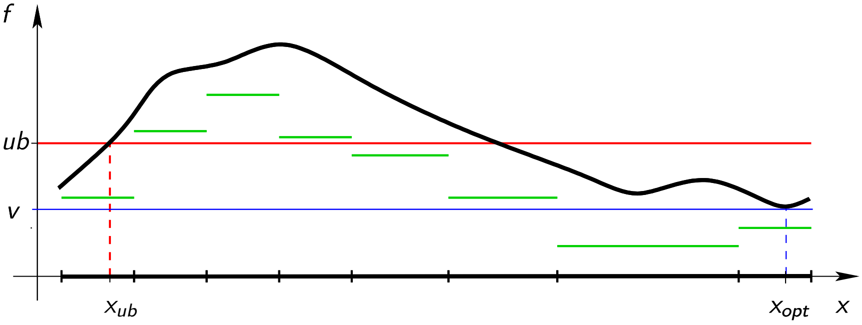
Subdivide M into smaller sets M' ...

Single objective B&B – partial lower bounds



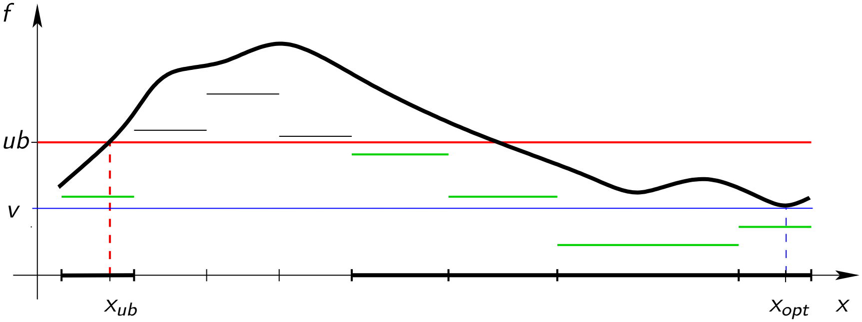
... and on each subset M' compute a partial lower bound $\ell b'$ for f.

Single objective B&B – the overall upper bound



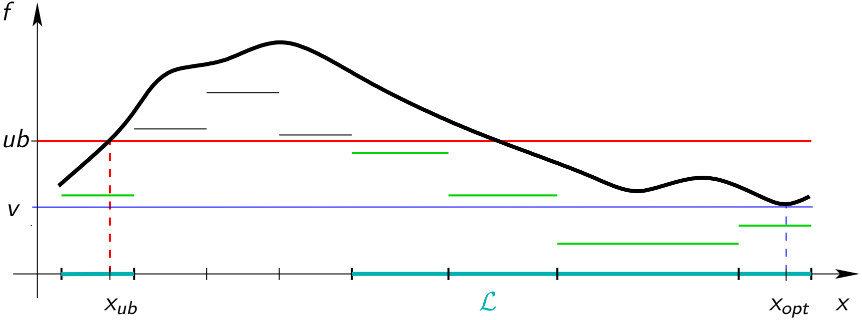
Any $x_{ub} \in M$ generates an overall upper bound ub for v ...

Single objective B&B – discarding / fathoming / pruning



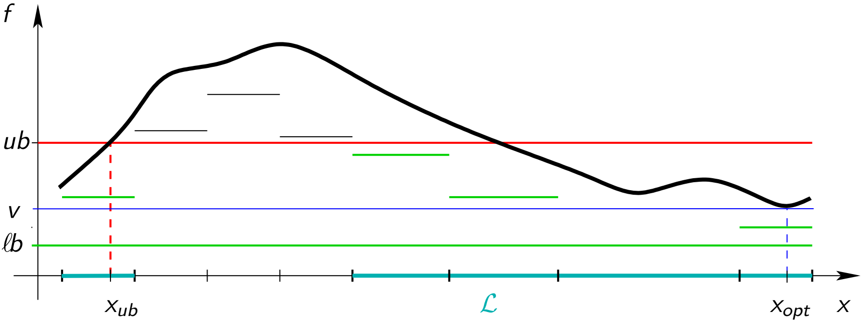
... so that all sets M' with $\ell b' > ub$ can be discarded ...

Single objective B&B – the list



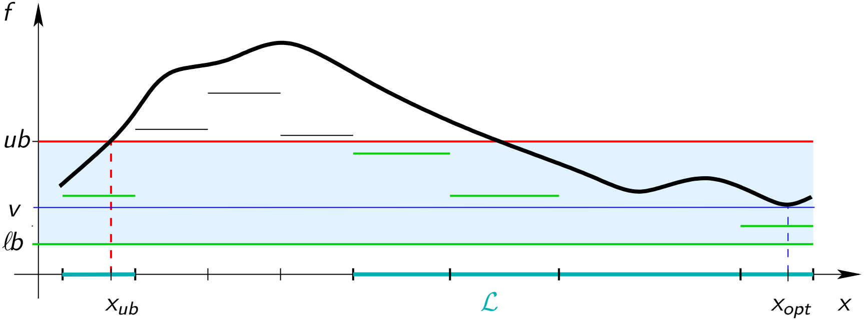
... and we only need to keep the list $\mathcal{L}$ of M' with $\ell b' \leq ub$.

Single objective B&B – the overall lower bound



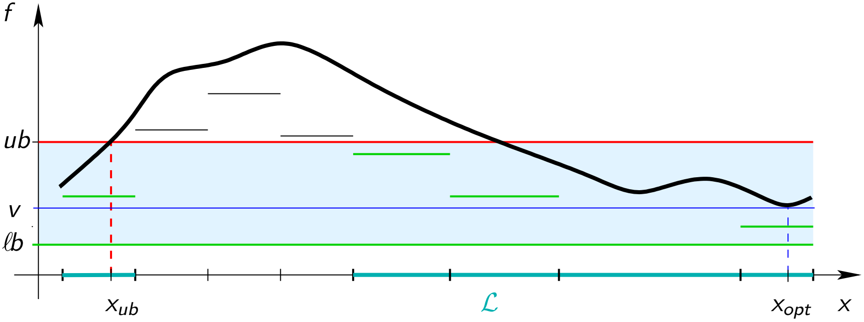
$\ell b \;:=\; \min_{M' \in \mathcal{L}} \ell b'$ is an overall lower bound for v.

Single objective B&B – termination criterion



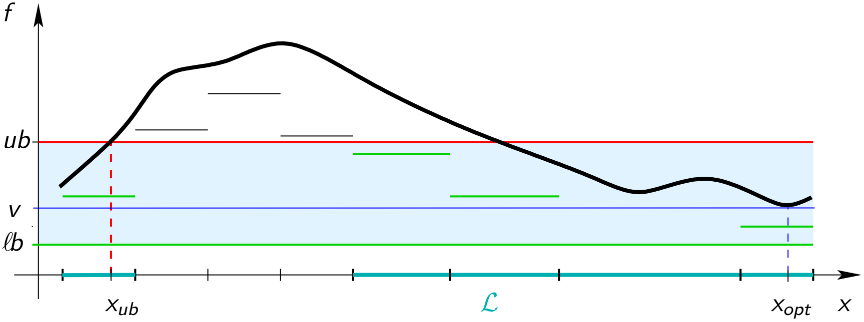
Common termination criterion: $ub - \ell b \; < \; \varepsilon$

Single objective B&B – ε-optimality



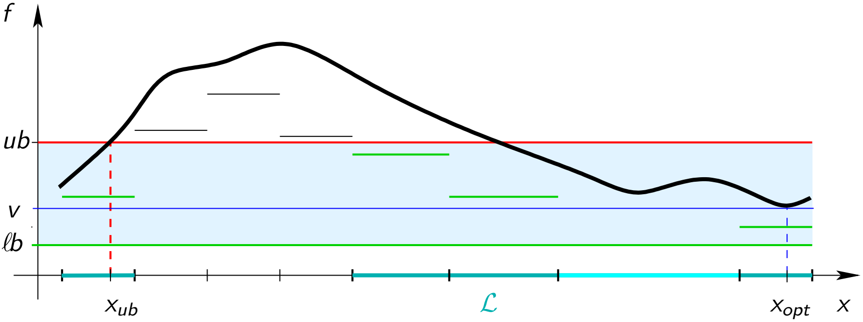
For $ub - \ell b \ < \ \varepsilon$ we have $v \ \leq \ f(x_{ub}) \ \leq \ v + \varepsilon$.

Single objective B&B – ε-optimality



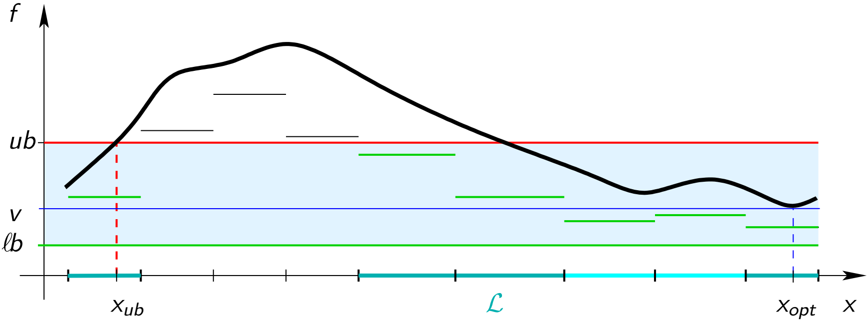
In particular, no $x \in M$ satisfies $f(x) < f(x_{ub}) - \varepsilon$.

Single objective B&B – node selection



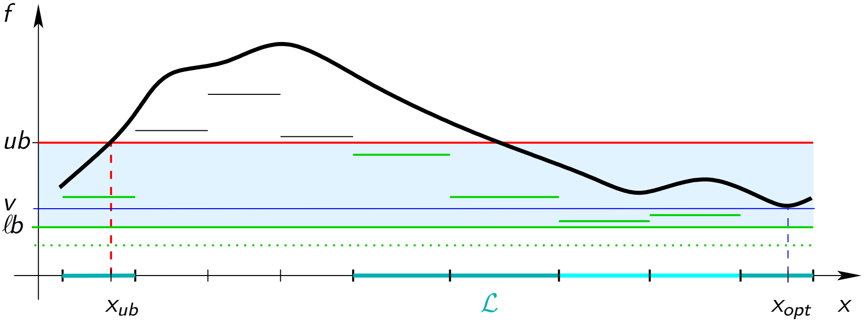
Choose some M' with $\ell b' = \ell b$ ...

Single objective B&B – branching



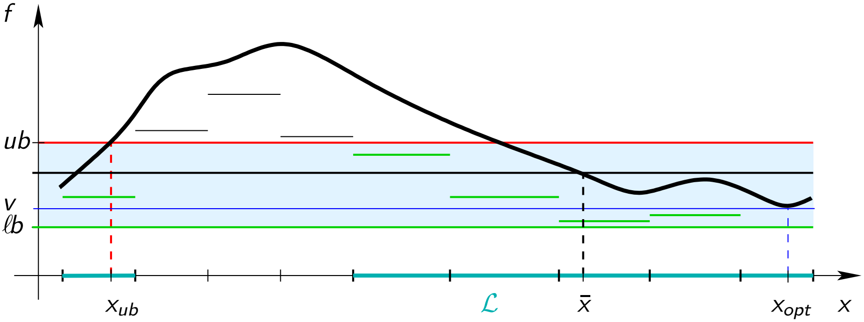
… branch it into two smaller sets, compute new lower bounds, …

Single objective B&B – improved overall lower bound



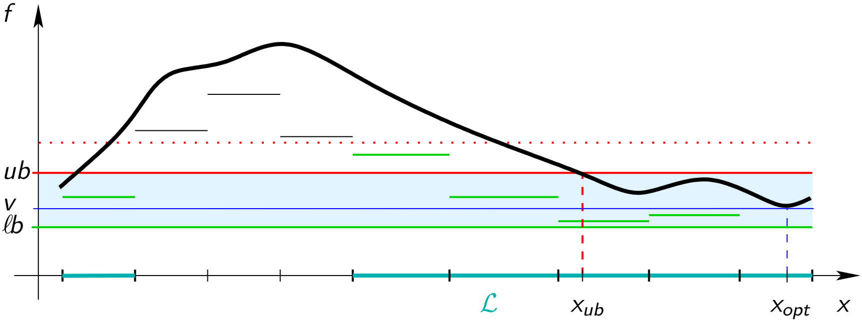
… and update ℓb.

Single objective B&B – improved upper bound



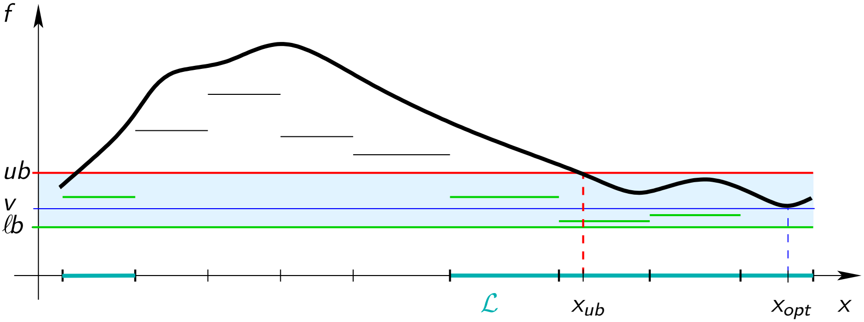
If along the way a better feasible point than x_{ub} is found, ...

Single objective B&B – update of the upper bound



… then also update x_{ub} and ub …

Single objective B&B – improved discarding



... and, if possible, discard further sets M' from $\mathcal{L}$.

Branching boxes

Branching is usually implemented by using

$$M \ = \ M(X) \ = \ \{x \in X|\ g(x) \le 0\},$$

just branching the box X by, e.g., halving it into X^1 and X^2,

and setting $M^1 := M(X^1)$, $M^2 := M(X^2)$.

B&B output

Upon termination we have

- $v \in [\ell b, ub]$ with $ub - \ell b < \varepsilon$,
- $\bigcup_{M' \in \mathcal{L}} M'$ covers the set of globally minimal points,

but we don't have

- the boxes X' with $M' = M(X')$ become small,
- the intervals $f(M')$ become small,
- $\bigcup_{M' \in \mathcal{L}} M'$ consists of ε-optimal points.

In particular, B&B focuses on the approximation of v,
but not of optimal points.

B&B output

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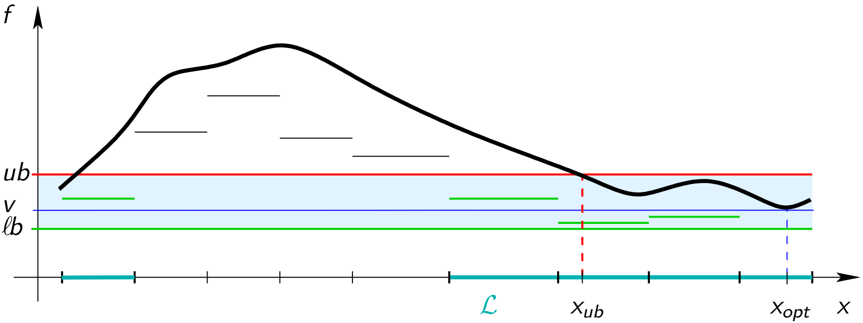
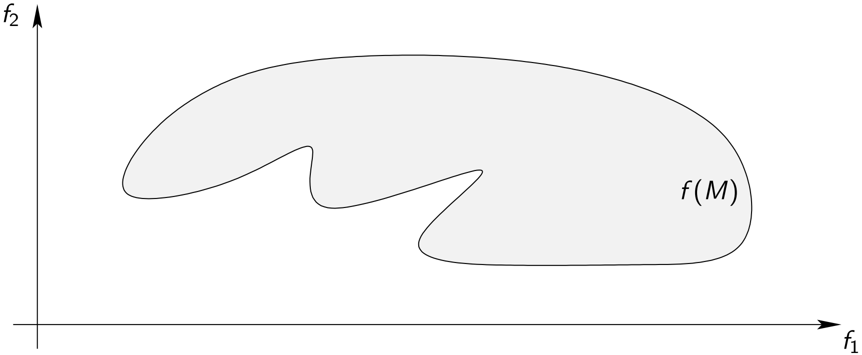
B&B output



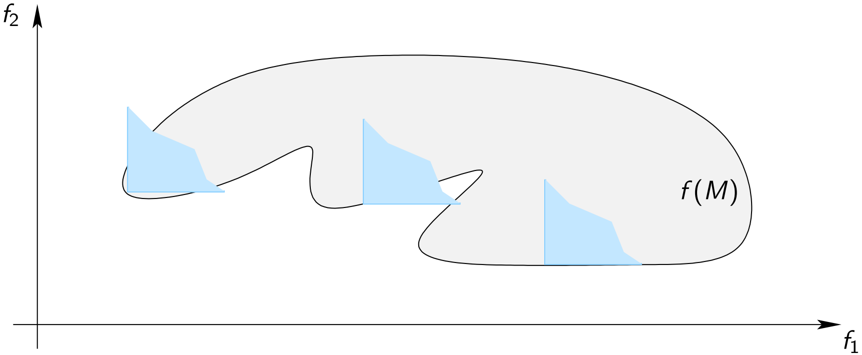
No small boxes,
$\bigcup_{M' \in \mathcal{L}} M'$ does not only consist of ε-optimal points.

Multiobjective optimality notions



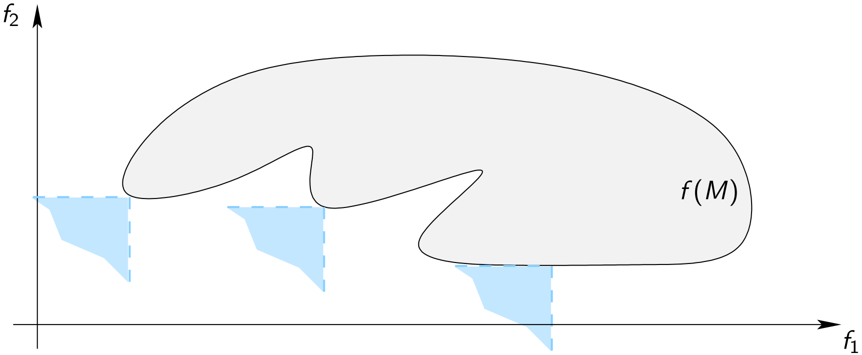
Appropriate generalization of ' $f(x) \geq f(\bar{x}) \;\; \forall\, x \in M$ ' ?

Multiobjective optimality notions



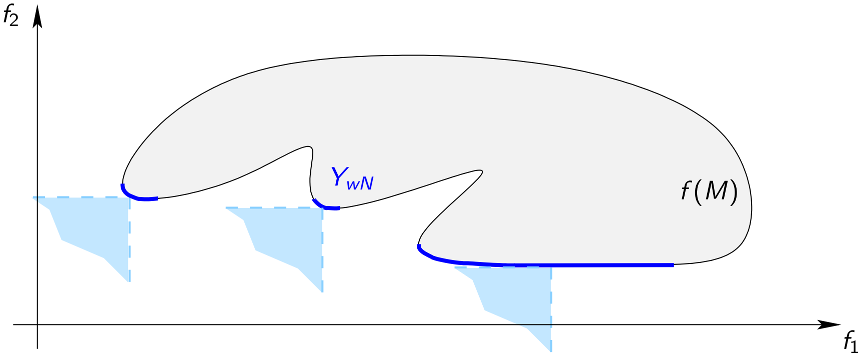
It is not '$f(x) \geq f(\bar{x}) \;\forall\, x \in M$'.

Multiobjective optimality notions



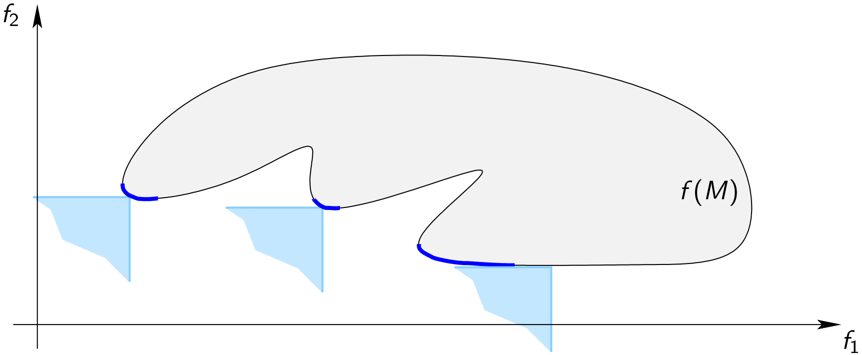
Better generalization: '$f(x) < f(\bar{x})$ for no $x \in M$'.

Multiobjective optimality notions



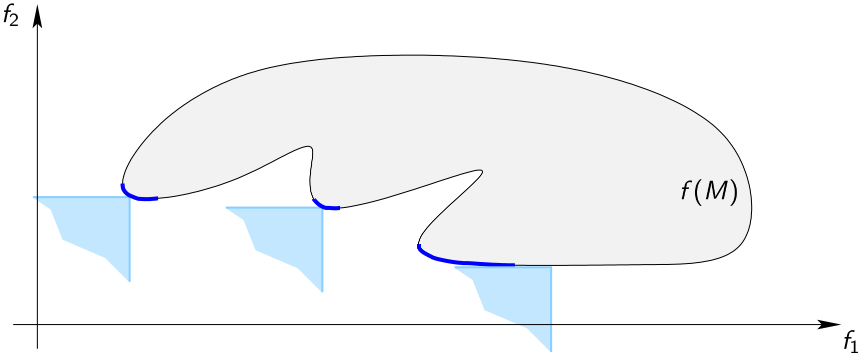
This leads to the weakly nondominated set Y_{wN}.

Multiobjective optimality notions



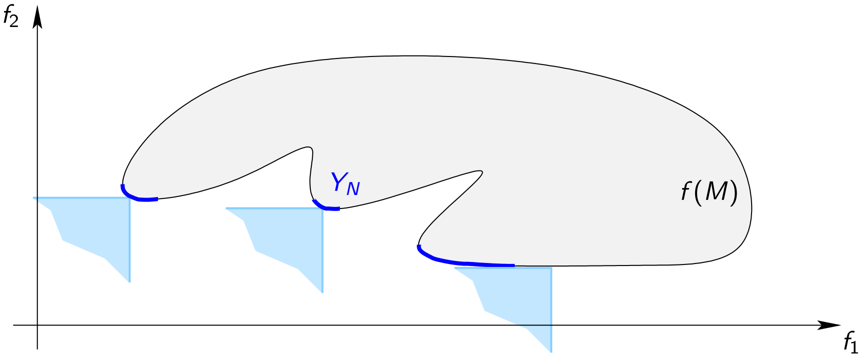
Even better: ' $f(x) \leq f(\bar{x}),\ f(x) \neq f(\bar{x})$ for no $x \in M$ '.

Multiobjective optimality notions



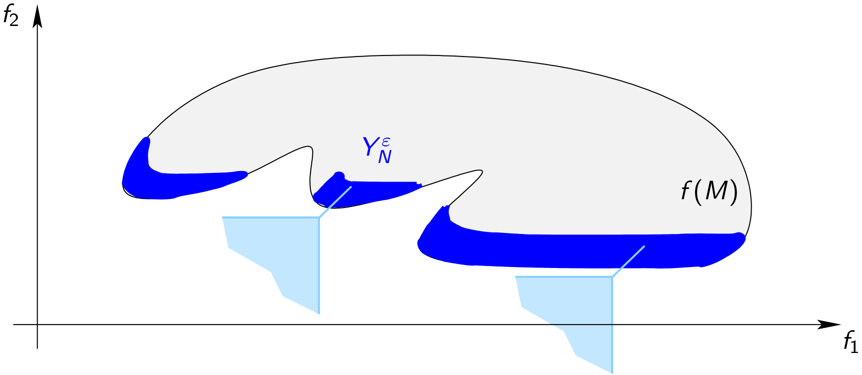
$\bar{x}$ is then called efficient, and $f(\bar{x})$ nondominated.

Multiobjective optimality notions



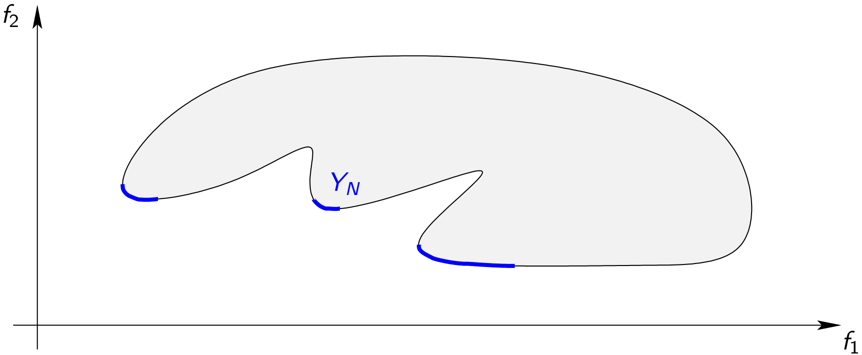
The nondominated set Y_N hence plays the role of v.

Multiobjective optimality notions



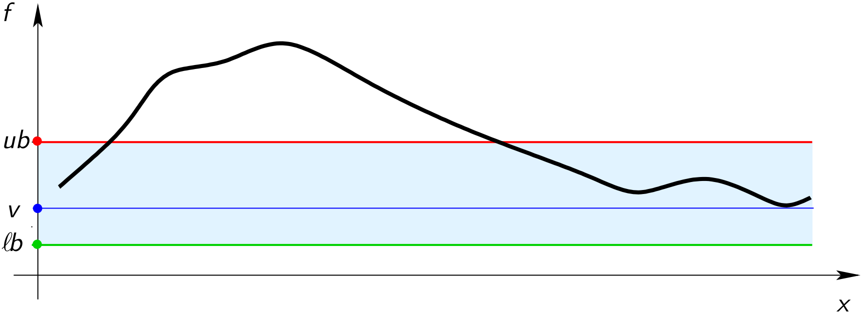
ε-efficiency: $f(x) \leq f(\bar{x}) - \varepsilon e,\ f(x) \neq f(\bar{x}) - \varepsilon e$ for no $x \in M$.

Enclosing Y_N



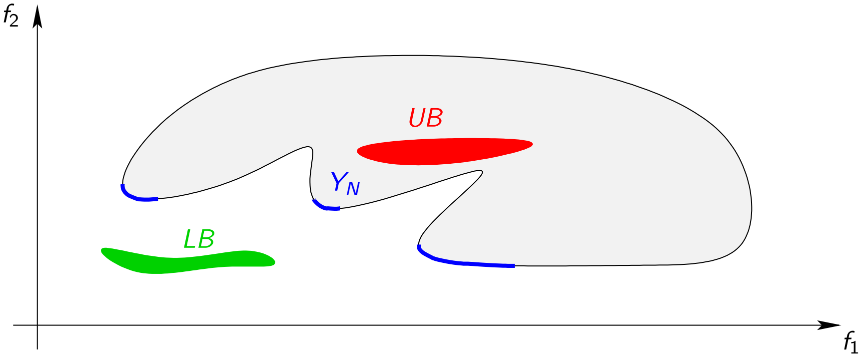
Generalization of the sandwiching property $\ell b \leq v \leq ub$?

Enclosing Y_N



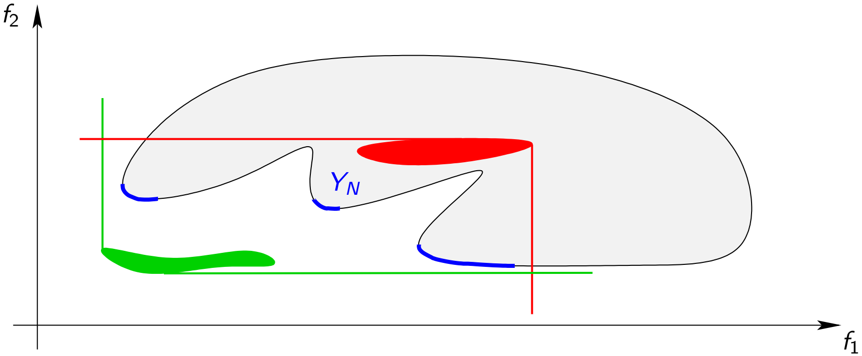
$$\ell b \leq v \leq ub \quad \Leftrightarrow \quad \{v\} \subseteq (\ell b + \mathbb{R}_+) \cap (ub - \mathbb{R}_+)$$

Enclosing Y_N



Let us construct nonempty and compact sets LB, UB with $Y_N \subseteq (LB + \mathbb{R}^m_+) \cap (UB - \mathbb{R}^m_+)$.

Enclosing Y_N



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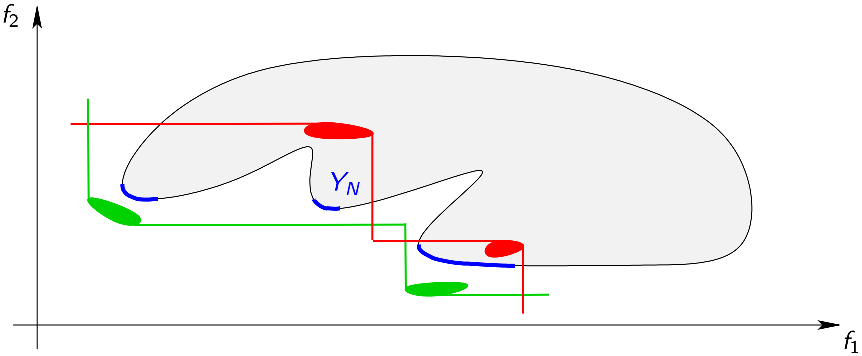

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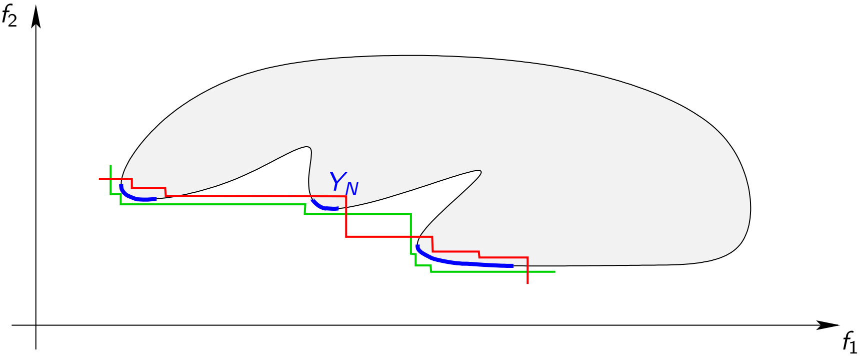

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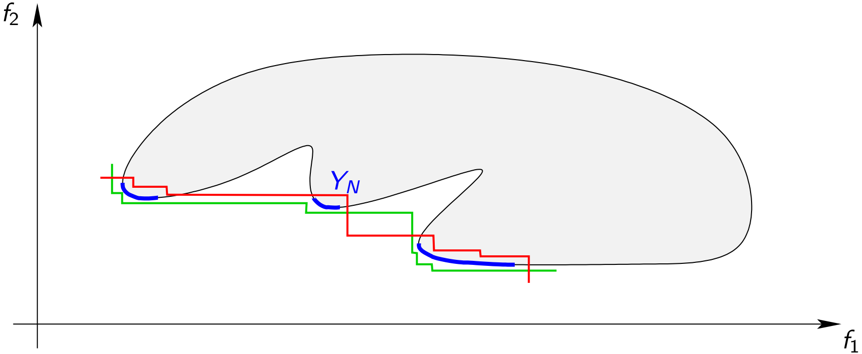
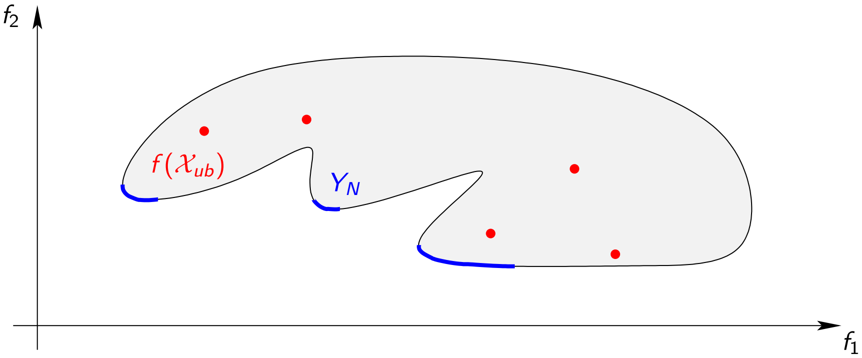


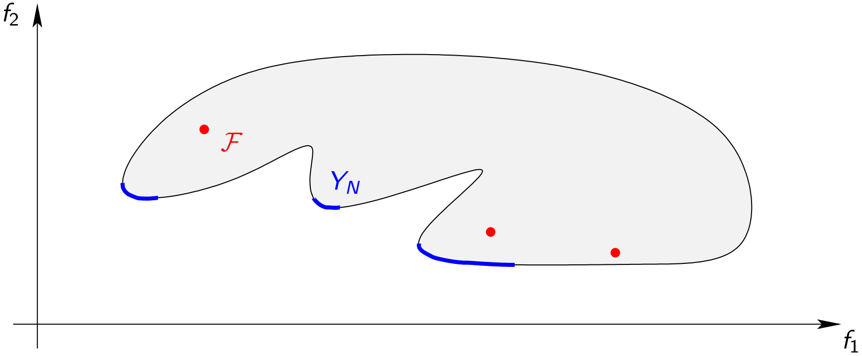
For some width measure $w(LB, UB)$ of $(LB + \mathbb{R}^m_+) \cap (UB - \mathbb{R}^m_+)$ the natural termination criterion then would be $w(LB, UB) < \varepsilon$.

The provisional nondominated set



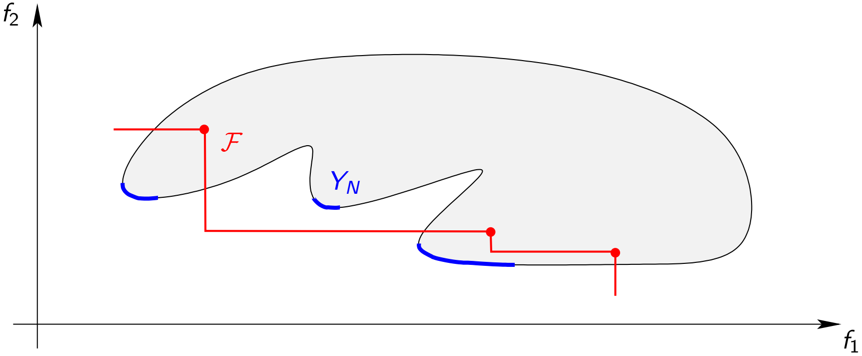
There is not a single best known feasible point with $ub = f(x_{ub})$, but a whole set $\mathcal{X}_{ub}$ and its image set $f(\mathcal{X}_{ub})$.

The provisional nondominated set



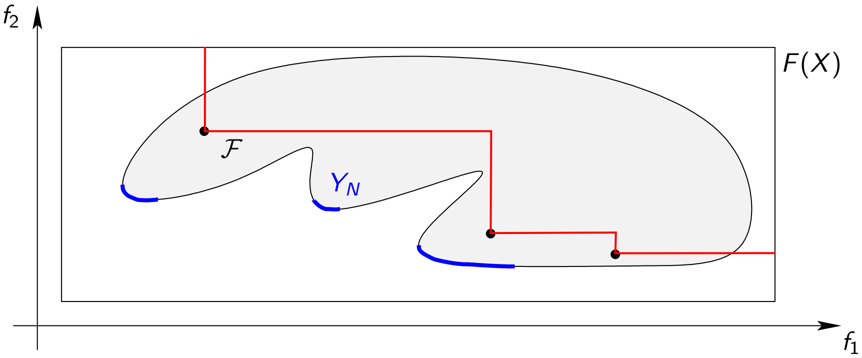
At least we can ignore the dominated points among $f(\mathcal{X}_{ub})$.
The remaining set $\mathcal{F}$ is the provisional nondominated set.

The provisional nondominated set



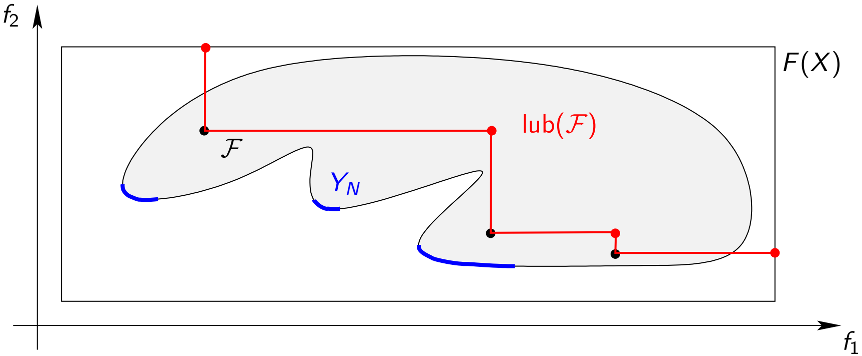
Unfortunately, $\mathcal{F} - \mathbb{R}^m_+$ does not necessarily contain Y_N, so that the choice $UB := \mathcal{F}$ is not possible.

The search region



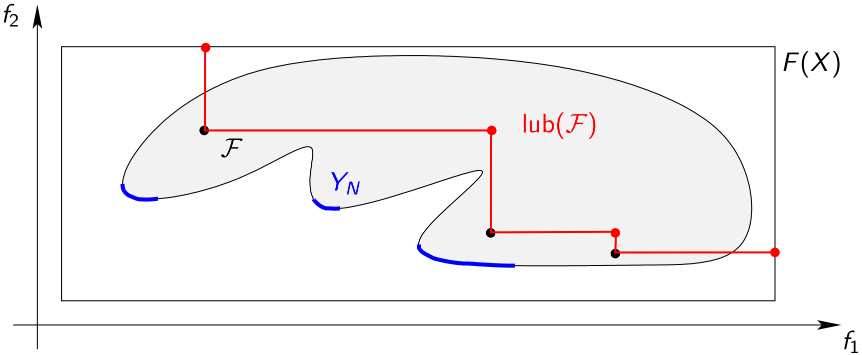
Instead, given $\mathcal{F}$ we consider the search region
of points which are not dominated by any point from $\mathcal{F}$.

Local upper bounds



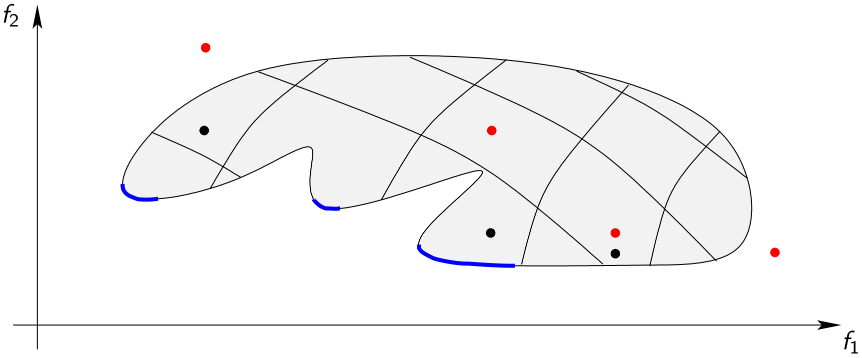
The search region may be described as $\text{lub}(\mathcal{F}) - \mathbb{R}^m_{++}$ with the computable finite set of local upper bounds $\text{lub}(\mathcal{F})$.

The upper bounding set



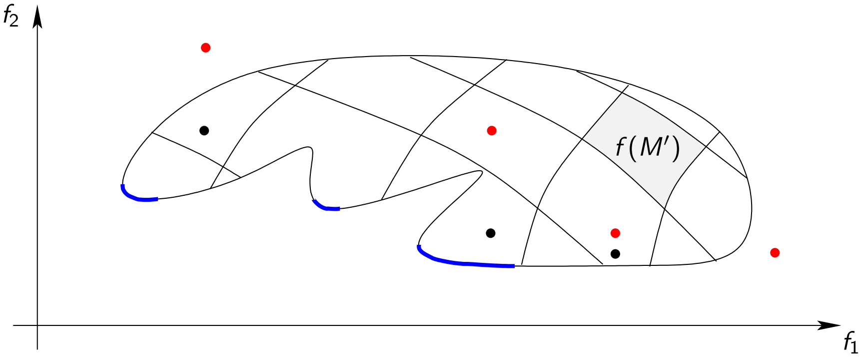
In view of $Y_N \subseteq \text{lub}(\mathcal{F}) - \mathbb{R}^m_+$ we may put $UB := \text{lub}(\mathcal{F})$.

Subdivision of $f(M)$



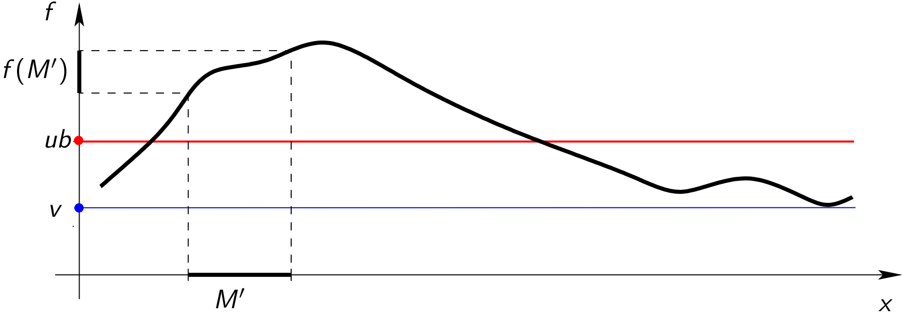
Subdivision of M induces subdivision of $f(M)$
(tesselation of M does not necessarily induce one of $f(M)$).

Discarding



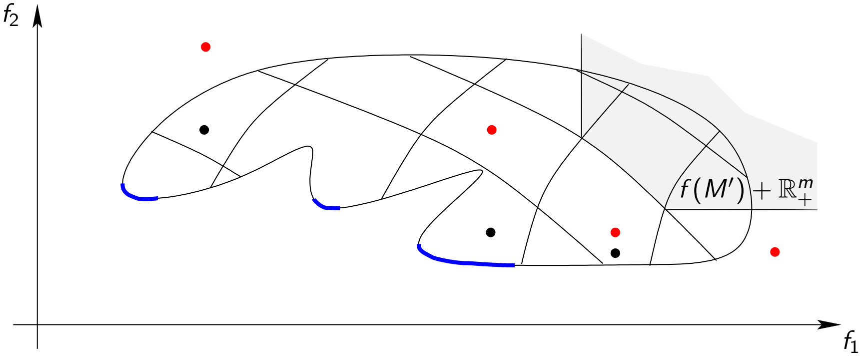
How to decide that M' can be discarded?

Discarding



$$ub < \min_{x \in M'} f(x) \quad \Leftrightarrow \quad \{ub\} \cap (f(M') + \mathbb{R}_+) = \emptyset$$

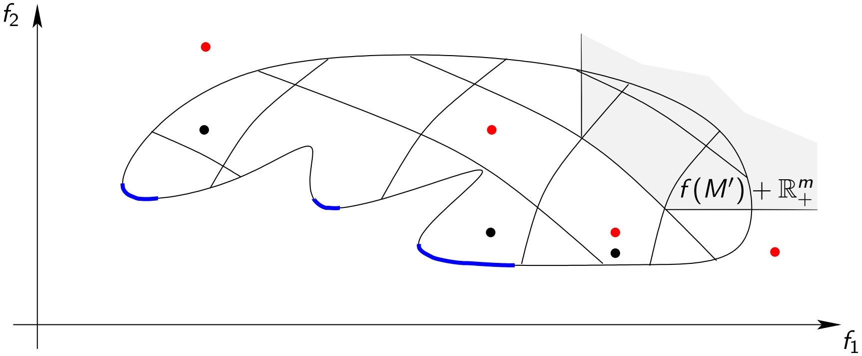
Discarding



Indeed: M' can be discarded if $\text{lub}(\mathcal{F}) \cap (f(M') + \mathbb{R}^m_+) = \emptyset$.

To check this, one needs a tractable description of $f(M') + \mathbb{R}^m_+$.

Discarding



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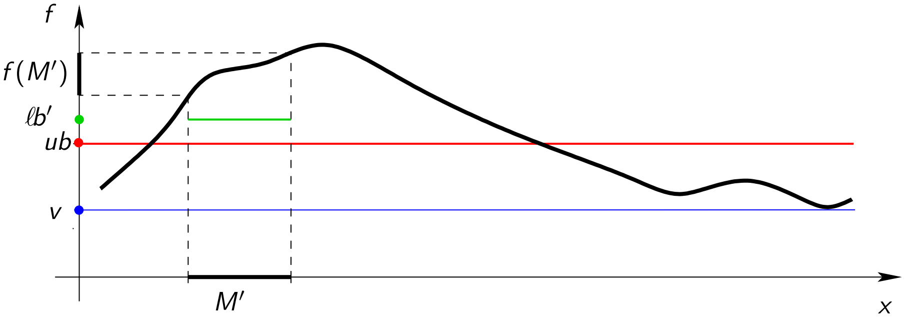
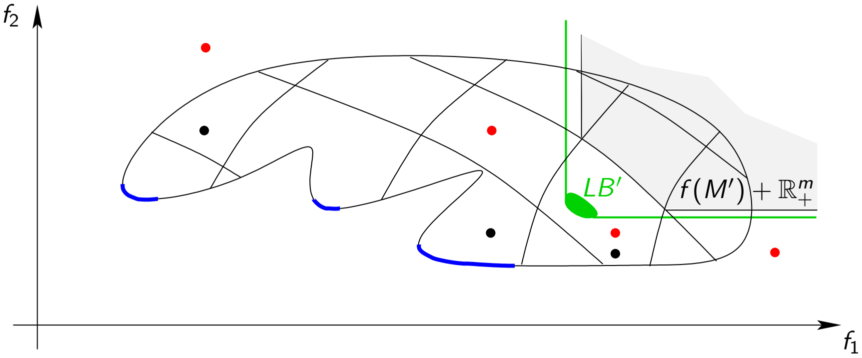
Discarding



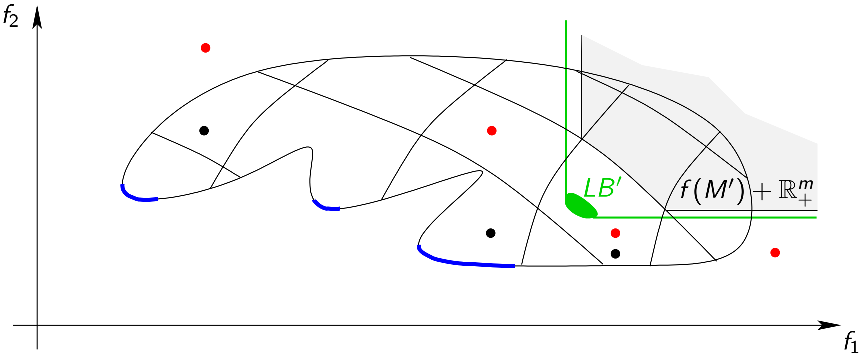
Discard M' if $ub < \ell b' \leq \min_{x \in M'} f(x)$
with the partial lower bound $\ell b'$.

Discarding



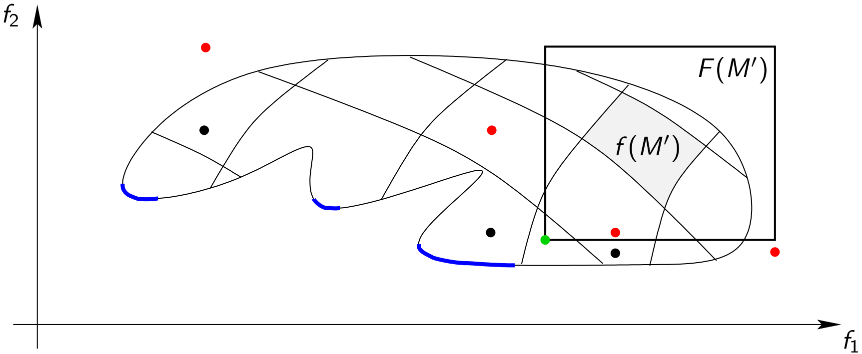
We call a compact set LB' with $f(M') + \mathbb{R}^m_+ \subseteq LB' + \mathbb{R}^m_+$ a partial lower bounding set for $f(M')$.

Discarding



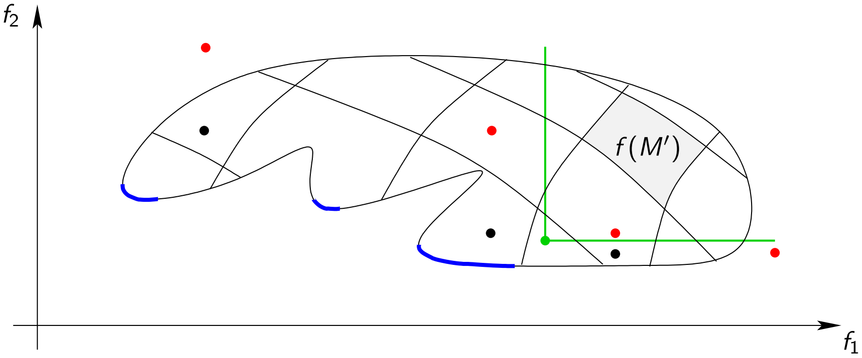
Hence M' can be discarded if $\text{lub}(\mathcal{F}) \cap (LB' + \mathbb{R}^m_+) = \emptyset$.

Discarding by interval arithmetic



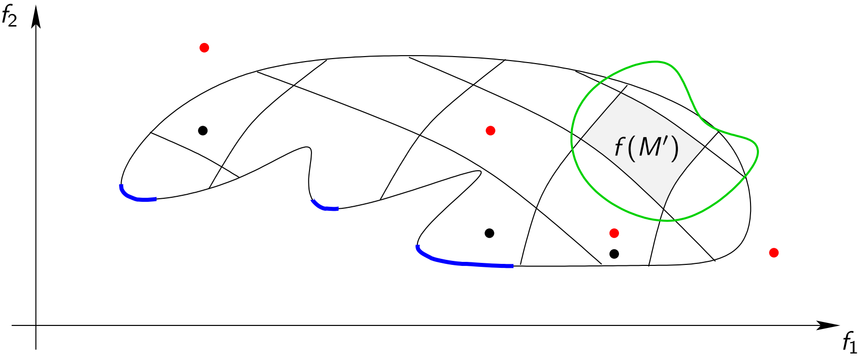
Sources of LB': interval arithmetic, ...

Discarding by interval arithmetic



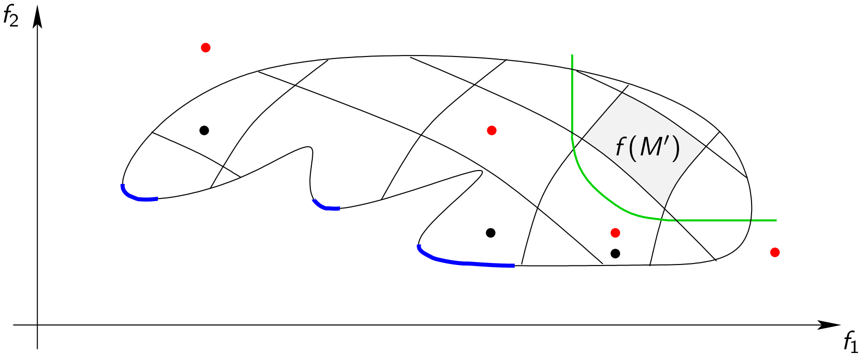
Sources of LB': interval arithmetic, ...

Discarding by convex relaxation



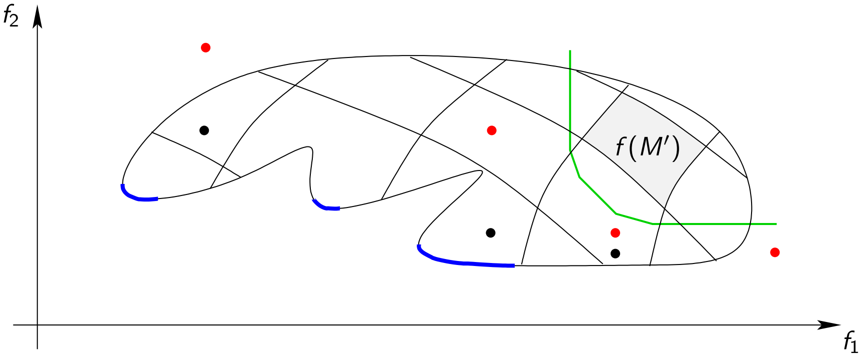
Sources of LB': convex relaxation, ...

Discarding by convex relaxation



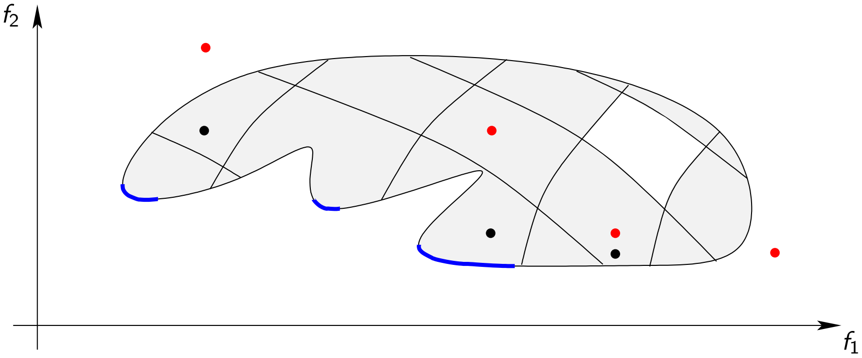
Sources of LB': convex relaxation, ...

Discarding by reformulation-linearization technique



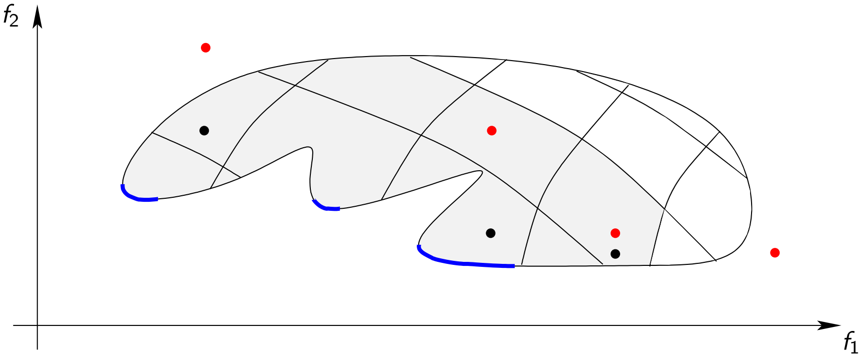
Sources of LB': RLT

Discarding



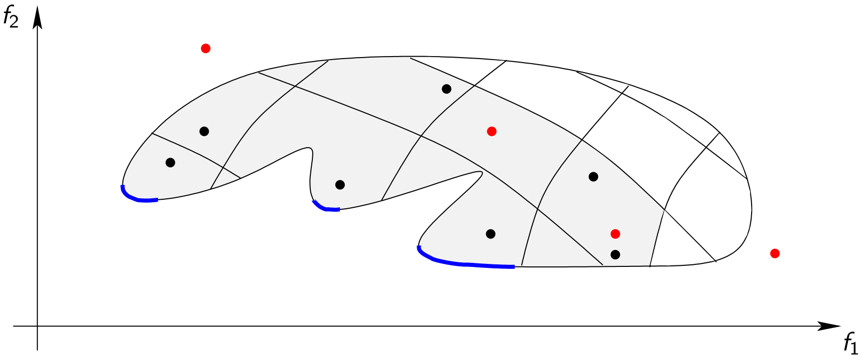
Say M' and its partial image set can be discarded, ...

Discarding



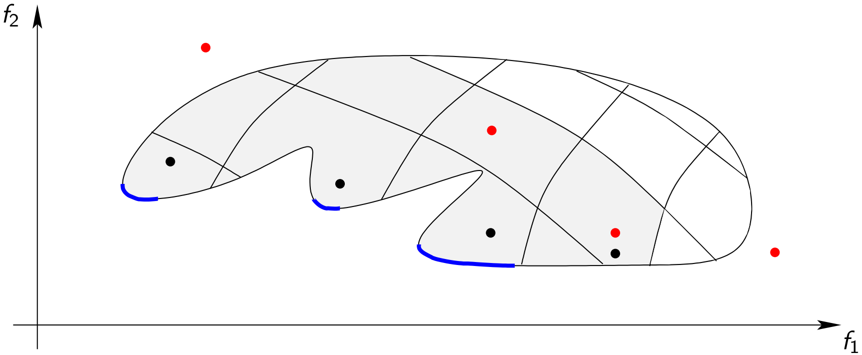
… as well as several other partial image sets.

New attainable points



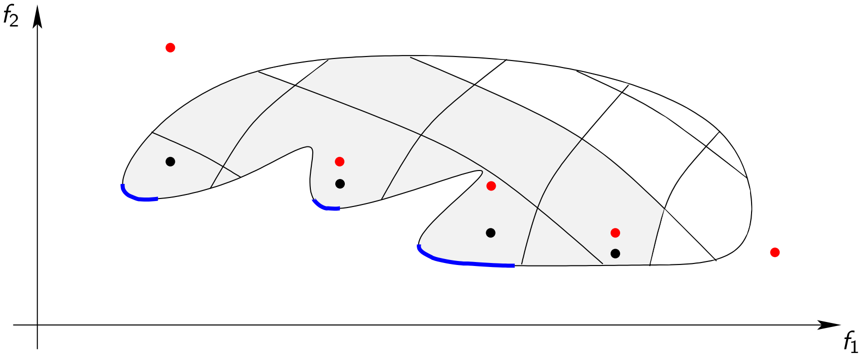
For the sets M' which have to be kept in $\mathcal{L}$, possibly new attainable points are computed during the failed discarding tests.

Update of $\mathcal{F}$



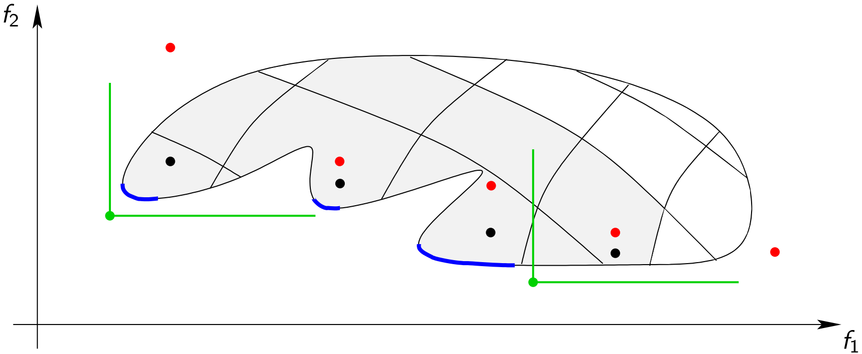
Then we update $\mathcal{F}$ ...

Update of lub($\mathcal{F}$) and $\mathcal{L}$



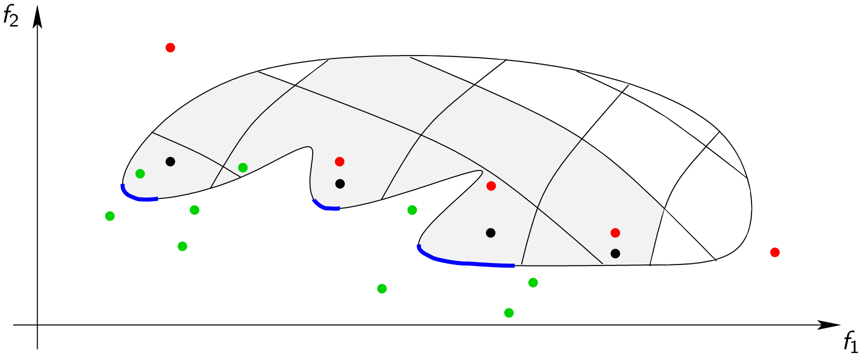
... as well as lub($\mathcal{F}$) and $\mathcal{L}$.

Partial lower bounds from $\mathcal{L}$



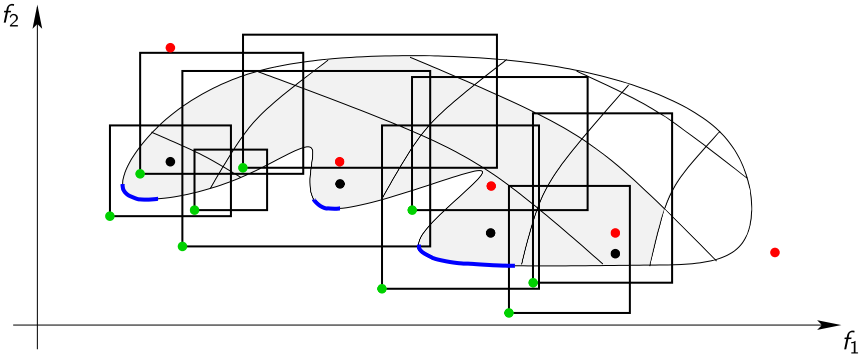
Each $M' \in \mathcal{L}$ is accompanied by some LB', say a singleton.

Partial lower bounds from $\mathcal{L}$



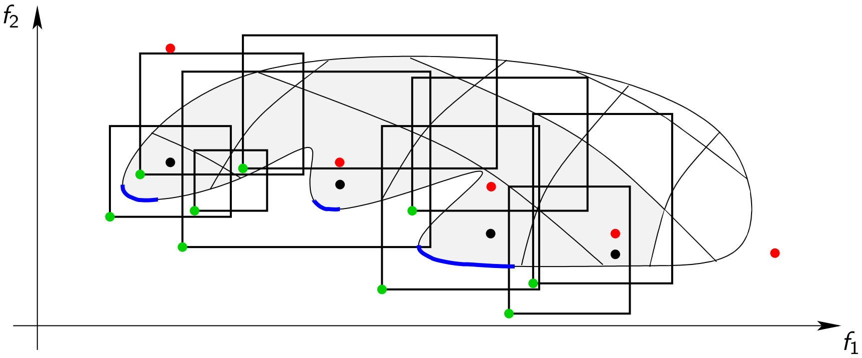
Each $M' \in \mathcal{L}$ is accompanied by some LB', say a singleton.

Enclosure of Y_N from the literature



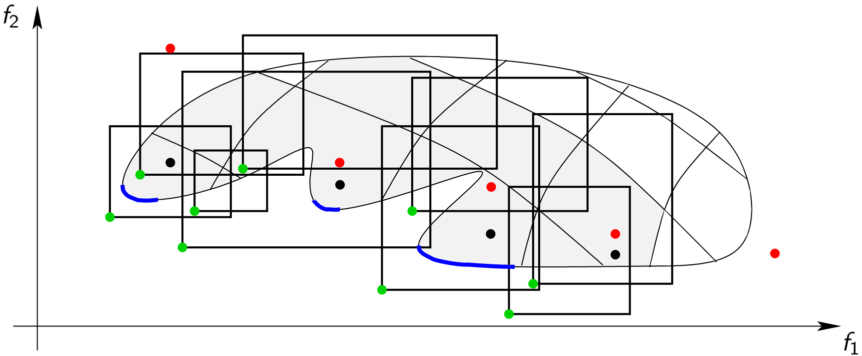
In the literature the enclosure $Y_N \subseteq \bigcup_{M' \in \mathcal{L}} F(M')$ is used, where $F(M')$ is some box with $f(M') \subseteq F(M')$.

Enclosure of Y_N from the literature



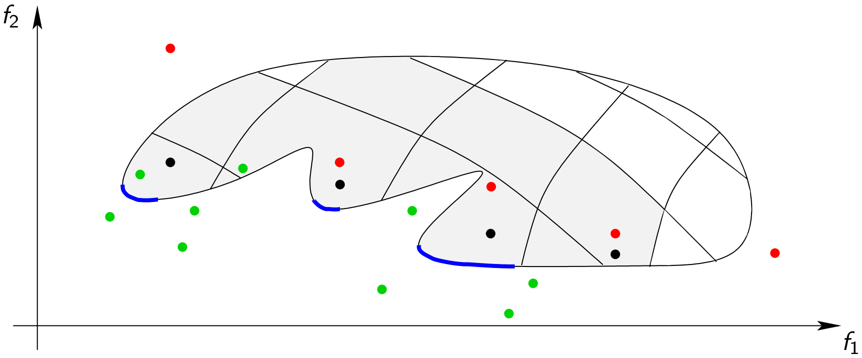
Common termination criterion: 'all boxes $F(M')$, $M' \in \mathcal{L}$, are small'
which is not consistent with single objective B&B.

Enclosure of Y_N from the literature



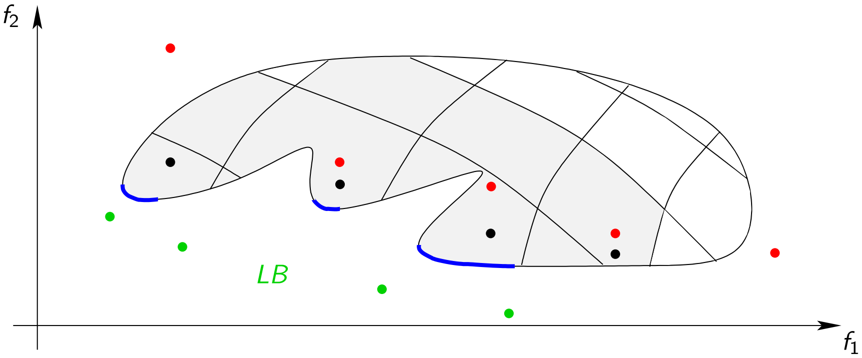
Common termination criterion: 'all boxes $F(M')$, $M' \in \mathcal{L}$, are small' which is not consistent with single objective B&B.

Overall lower bounding set



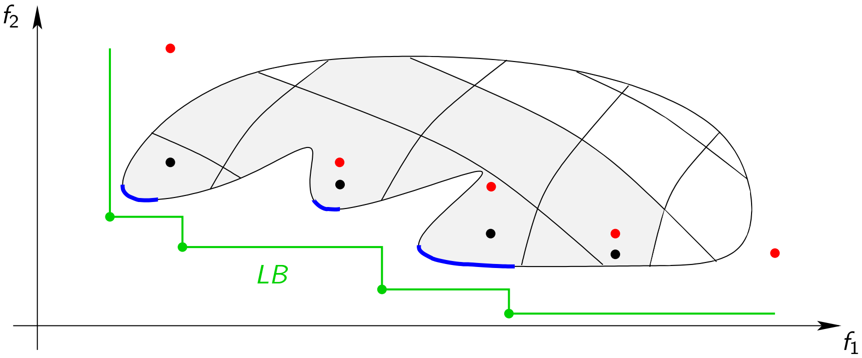
While for an overall lower bound the choice $LB := \bigcup_{M' \in \mathcal{L}} LB'$ is possible, many LB' in this union are redundant.

Overall lower bounding set



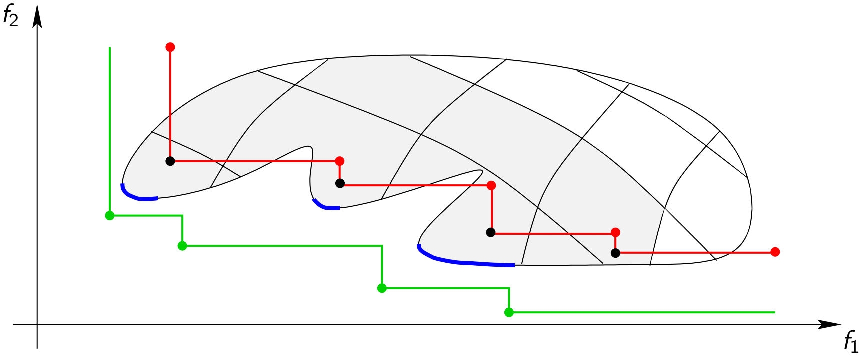
Instead we consider the sublist $\mathcal{L}_N$ of $M' \in \mathcal{L}$ such that LB' is nondominated and define $LB := \bigcup_{M' \in \mathcal{L}_N} LB'$.

Overall lower bounding set



This is in analogy to setting $\ell b := \min_{M' \in \mathcal{L}} \ell b'$ in the single objective case.

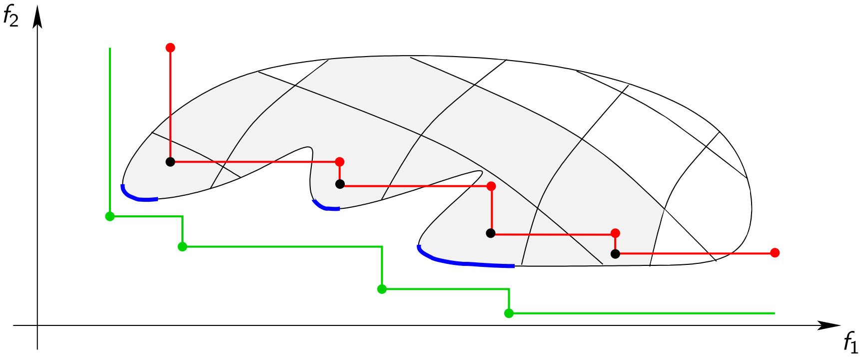
The sandwich



It results in the desired enclosure

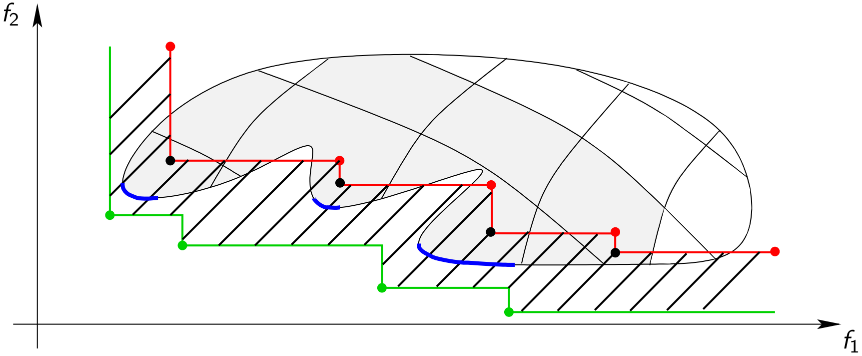
$$Y_N \subseteq (LB + \mathbb{R}^m_+) \cap (\text{lub}(\mathcal{F}) - \mathbb{R}^m_+) =: E(LB, \text{lub}(\mathcal{F})).$$

The sandwich



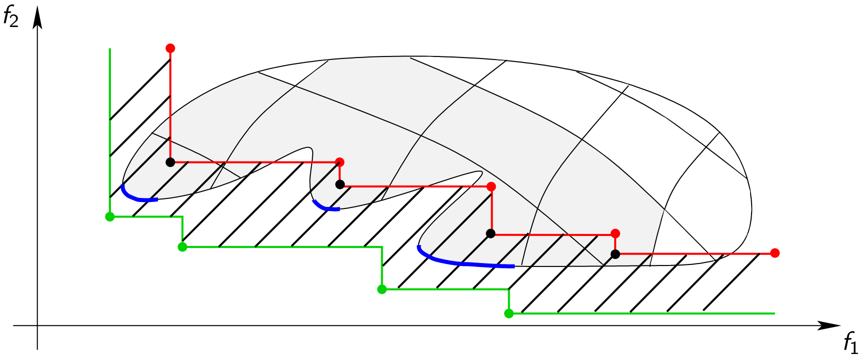
Theorem 1: $Y_N \cup \mathcal{F} \subseteq E(LB, \text{lub}(\mathcal{F}))$.

The sandwich width (geometrical definition)



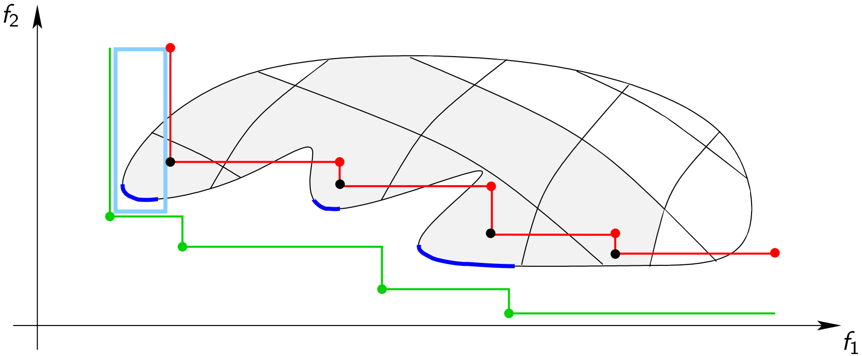
Let us measure the width of $w(LB, \mathrm{lub}(\mathcal{F}))$ of $E(LB, \mathrm{lub}(\mathcal{F}))$ with respect to the direction e, ...

The sandwich width (geometrical definition)



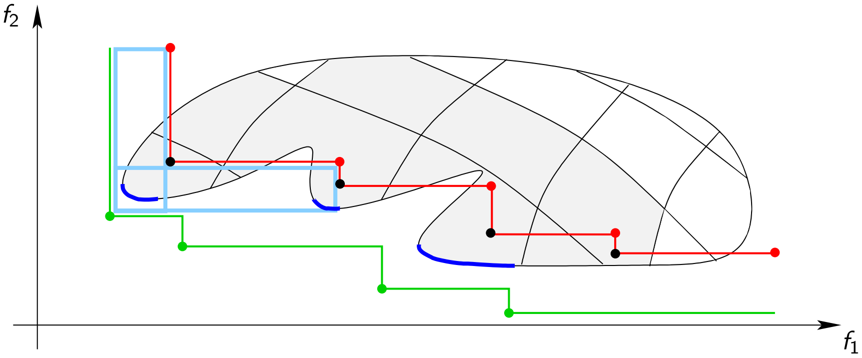
$$w(LB, \mathrm{lub}(\mathcal{F})) := \max\{\|(y+te)-y\|_2/\sqrt{m} \mid t \geq 0,\ y, y+te \in E(LB, \mathrm{lub}(\mathcal{F}))\}.$$

The sandwich boxes



The enclosure $E(LB, \text{lub}(\mathcal{F}))$ may be written as the union of boxes $\bigcup \{[a, p] \mid a \in LB,\ p \in \text{lub}(\mathcal{F}),\ a \leq p\}$.

The sandwich boxes



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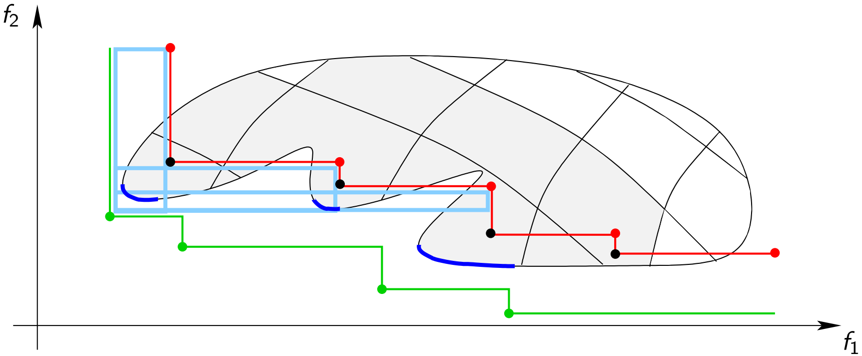
The sandwich boxes



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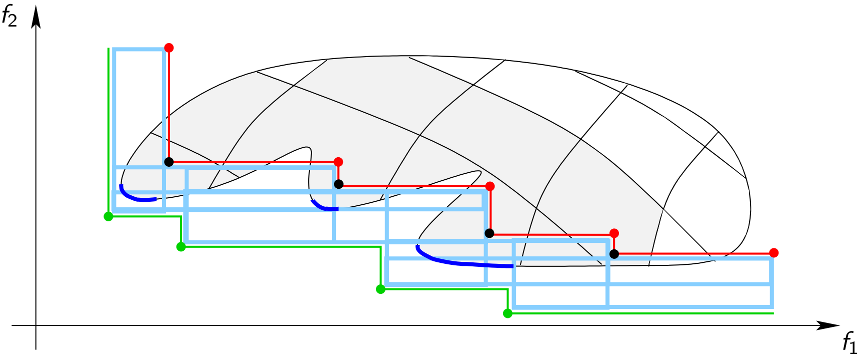
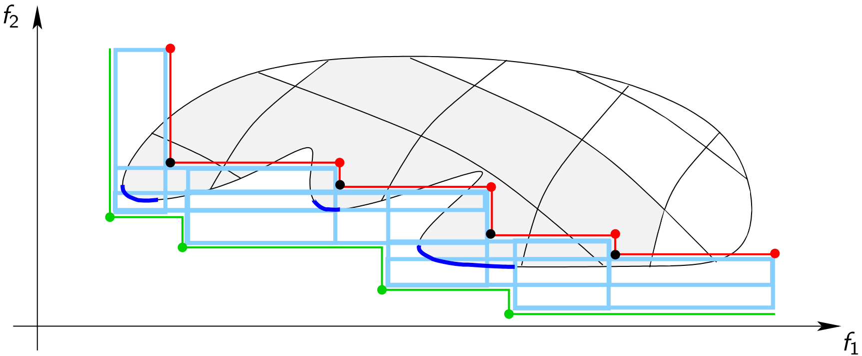
The sandwich boxes



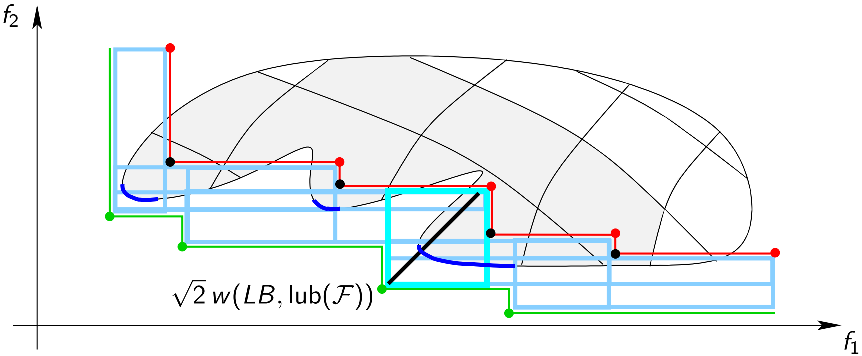
The enclosure $E(LB, \text{lub}(\mathcal{F}))$ may be written as the union of boxes $\bigcup \{[a, p] \mid a \in LB,\ p \in \text{lub}(\mathcal{F}),\ a \leq p\}$.

The sandwich boxes



For a box $[a, p]$ let $s(a, p) := \min_{j=1,\ldots,m}(p_j - a_j)$ denote the shortest edge length.

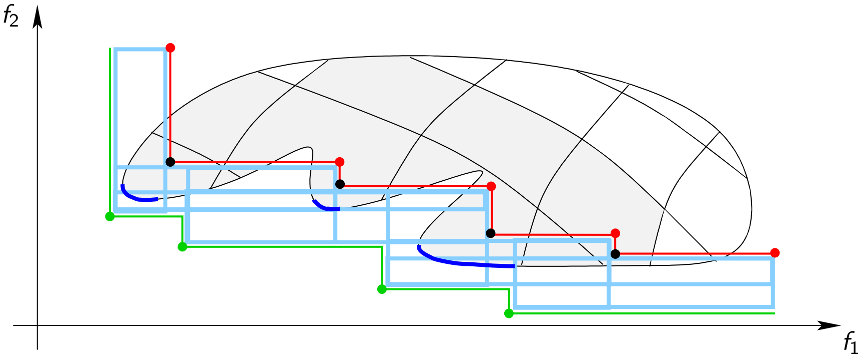
The sandwich width (tractable formula)



Lemma:

$$w(LB, \mathrm{lub}(\mathcal{F})) \;=\; \max\{s(a,p) \mid a \in LB,\ p \in \mathrm{lub}(\mathcal{F}),\ a \le p\}$$

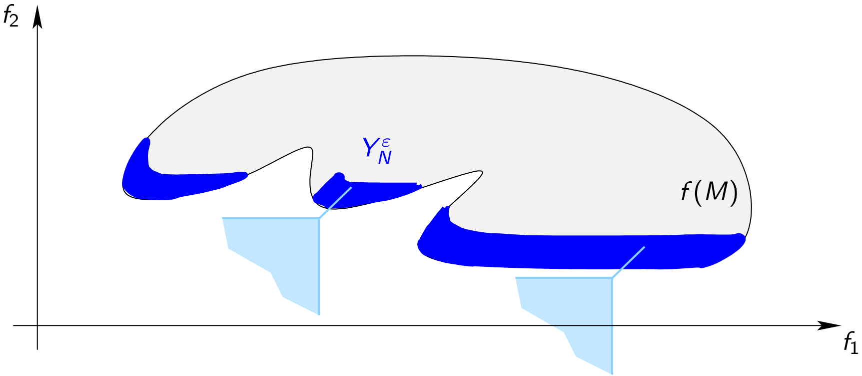
ε-nondominated $\mathcal{F}$



Theorem 2:

$\varepsilon > w(LB, \text{lub}(\mathcal{F})) \Rightarrow$ all $q \in \mathcal{F}$ are ε-nondominated.

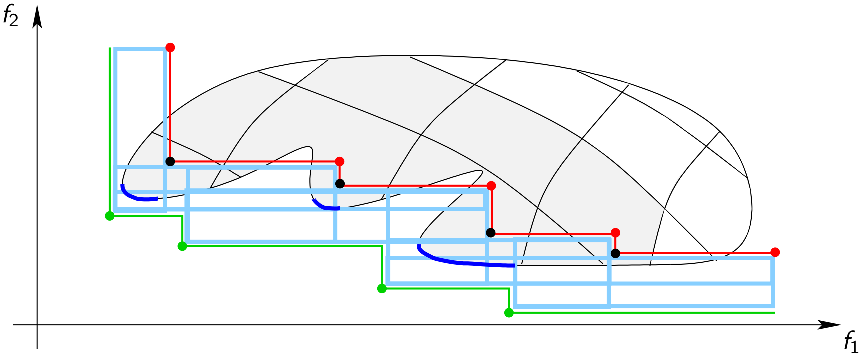
ε-nondominated $\mathcal{F}$



Theorem 2:

$\varepsilon > w(LB, \mathrm{lub}(\mathcal{F})) \quad \Rightarrow \quad$ all $q \in \mathcal{F}$ are ε-nondominated.

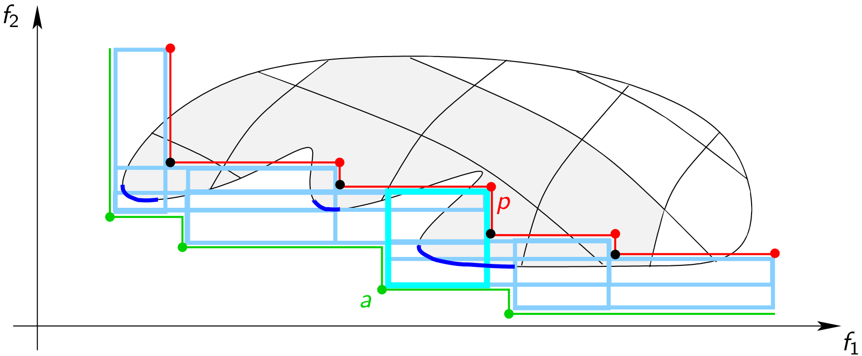
Width based termination criterion



Theorem 2 (termination criterion):

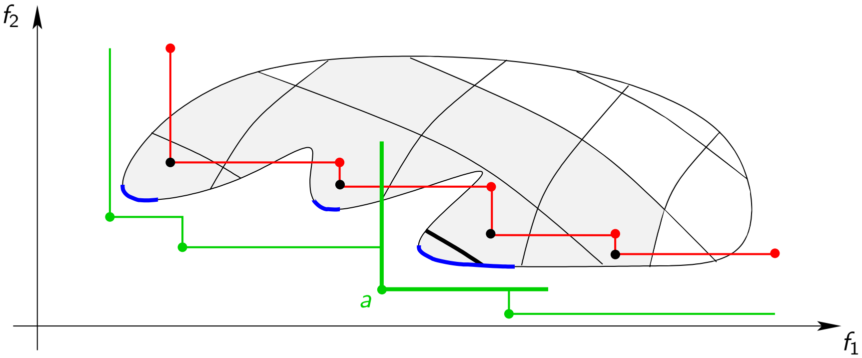
$\varepsilon \; > \; w(LB, \text{lub}(\mathcal{F})) \quad \Rightarrow \quad$ all $q \in \mathcal{F}$ are ε-nondominated.

Choosing a set to branch



Node selection: Choose some sandwich box $[a, p]$ with $s(a, p) = w(LB, \text{lub}(\mathcal{F}))$ and branch the set M'_a with partial lower bounding set $\{a\}$.

Choosing a set to branch



Node selection: Choose some sandwich box $[a, p]$ with
$s(a, p) = w(LB, \mathrm{lub}(\mathcal{F}))$
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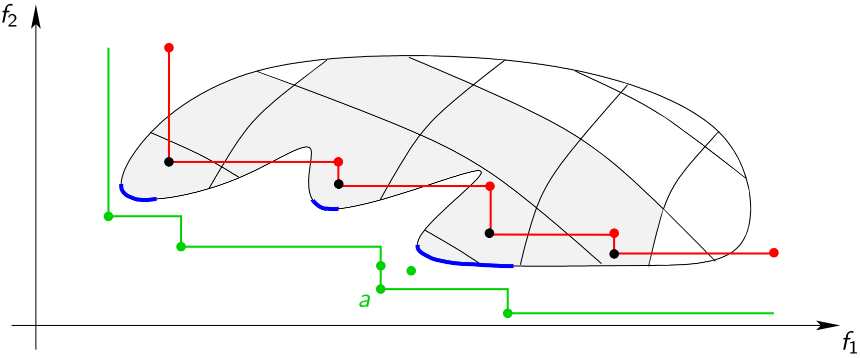
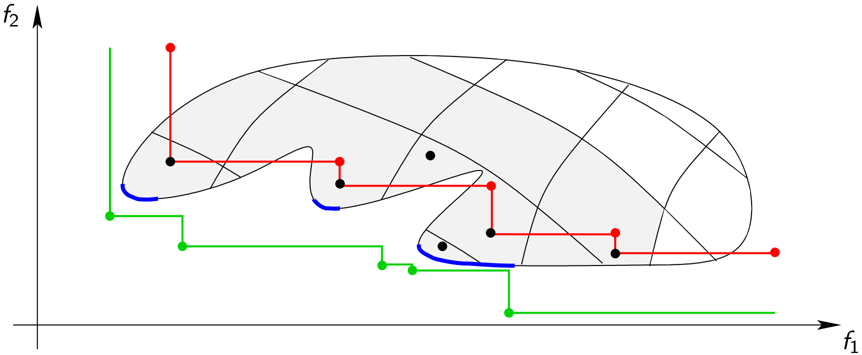
Branching step



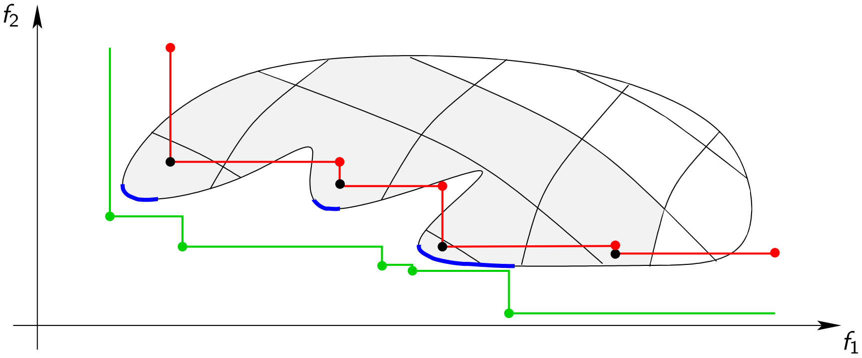
Compute new partial lower bounding sets, ...

Branching step



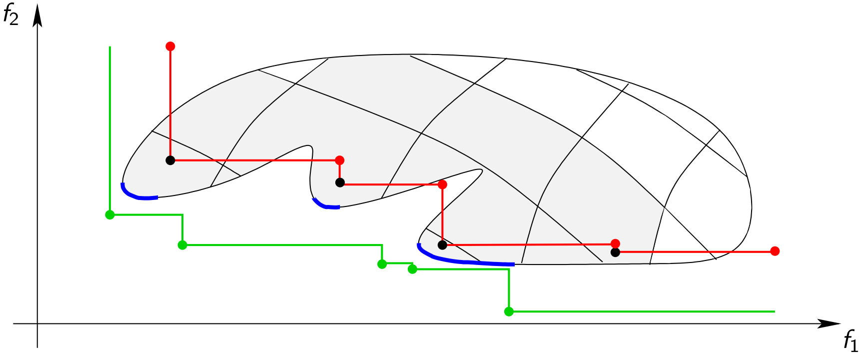
… try to discard, obtain new attainable points, update LB, …

Branching step



... update $\mathcal{F}$, update lub($\mathcal{F}$).

Convergence



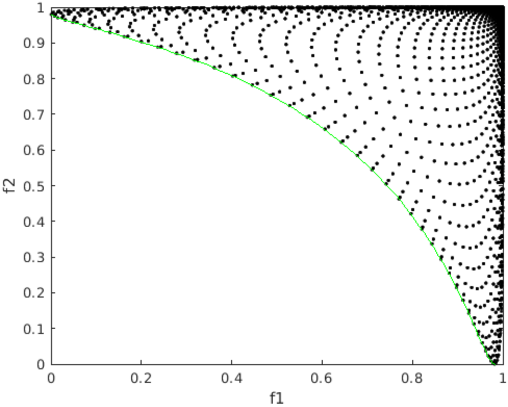
Theorem 3: Lower bounding by IA, αBB, RLT, or other convergent procedures + computability of feasible points if they exist $\Rightarrow$ for any ε B&B terminates after finitely many iterations.

Example 1 – Fonseca-Fleming problem

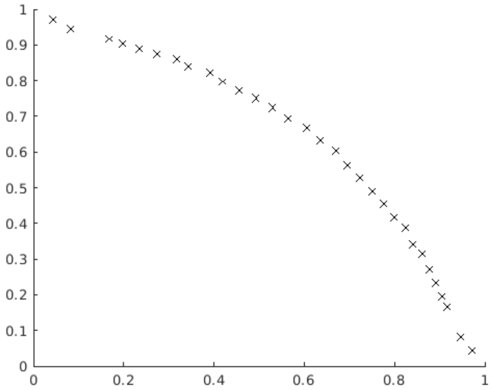
$$FF: \quad \min \begin{pmatrix} 1 - \exp(-(x_1 - 1/\sqrt{2})^2 - (x_2 - 1/\sqrt{2})^2) \\ 1 - \exp(-(x_1 + 1/\sqrt{2})^2 - (x_2 + 1/\sqrt{2})^2) \end{pmatrix}$$

$$\text{s.t.} \quad -4 \leq x_1, x_2 \leq 4.$$

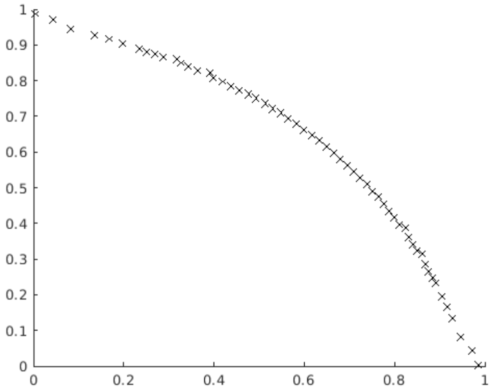
Example 1 – Attainable points



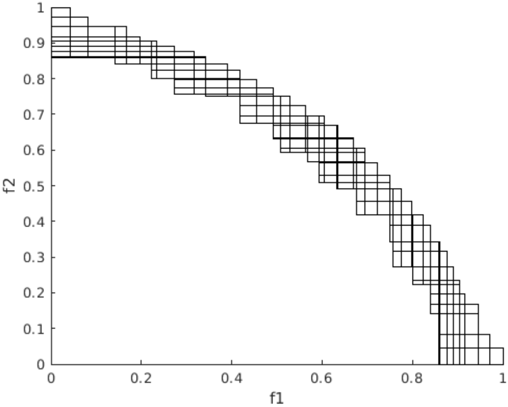
Example 1 – Provisional nondominated set, $\varepsilon = 0.1$



Example 1 – Provisional nondominated set, $\varepsilon = 0.05$



Example 1 – Enclosure, $\varepsilon = 0.1$



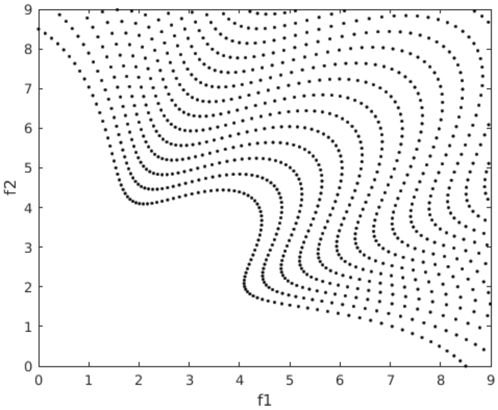
Example 2 – DEB2DK

$$DEB2DK: \quad \min \begin{pmatrix} r(x)\sin(x_1\pi/2) \\ r(x)\cos(x_1\pi/2) \end{pmatrix}$$

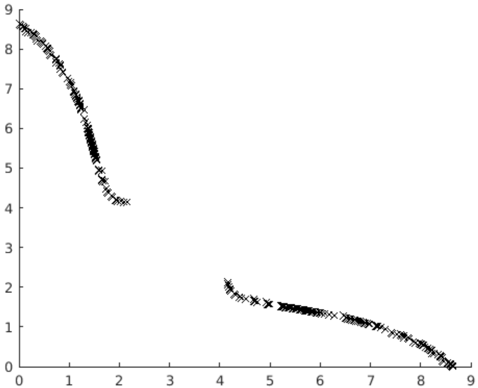
$$\text{s.t.} \quad 0 \le x_1, x_2 \le 1$$

$$\text{with} \quad r(x) = \left(5 + 10(x_1 - 0.5)^2 + \cos(4\pi x_1)\right)(1 + 9x_2).$$

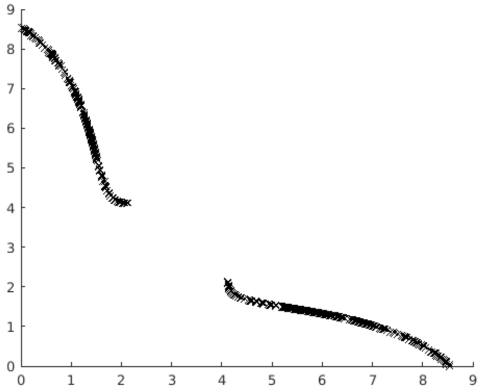
Example 2 – Attainable points



Example 2 – Provisional nondominated set, $\varepsilon = 0.1$



Example 2 – Provisional nondominated set, $\varepsilon = 0.05$



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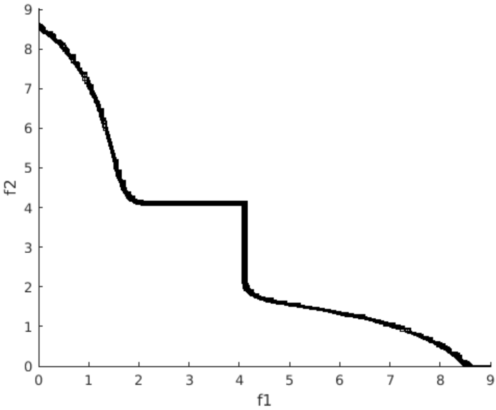
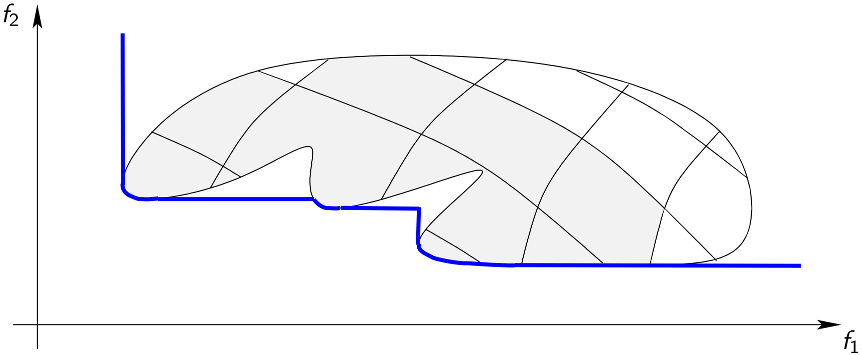


Open questions



Y_N is in general disconnected, while $E(LB, \text{lub}(\mathcal{F}))$ seems to converge to the seemingly connected weakly nondominated set of $f(M) + \mathbb{R}^m_+$.

Further literature

K. Dächert, K. Klamroth, R. Lacour, D. Vanderpooten, *Efficient computation of the search reagion in multi-objective optimization*, European Journal of Operational Research, Vol. 260 (2017), 841–855.

M. Ehrgott, *Multicriteria Optimization*, Springer, 2005.

K. Klamroth, R. Lacour, D. Vanderpooten, *On the representation of the search region in multi-objective optimization*, European Journal of Operational Research, Vol. 245 (2015), 767–778.

K. Miettinen, *Nonlinear Multiobjective Optimization*, Springer, 1998.