Lifting for simplicity: concise descriptions of convex sets

James Saunderson

Electrical and Computer Systems Engineering,
Monash University, Australia

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Joint work with:

Hamza Fawzi
U. Cambridge

João Gouveia
U. Coimbra

Pablo Parrilo
MIT

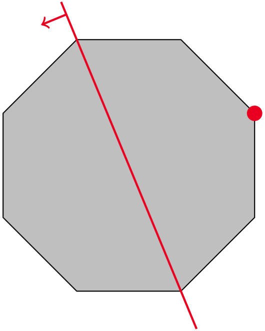
Rekha Thomas
U. Washington

Linear programming

Minimization of linear functional over polyhedron

Linear program:

$$\min_{x} \ \langle c, x \rangle$$
$$\text{s.t. } Ax = b, \quad x \geq 0$$



Polyhedron: affine slice of some non-negative orthant

m facets: affine slice of $\mathbb{R}^m_+$

Linear programming

Minimization of linear functional over polyhedron with not too many facets

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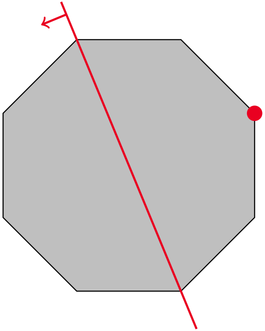


Polyhedron: affine slice of some non-negative orthant

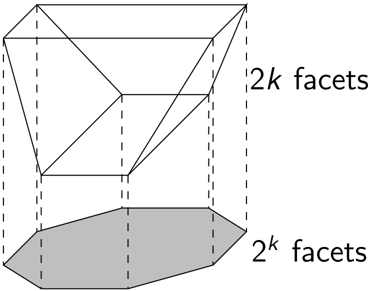
m facets: affine slice of $\mathbb{R}^m_+$

What is the simplest description of an octagon
for linear programming?

Linear programming? Projection helps

Minimization of linear functional over projection of polyhedron with not too many facets

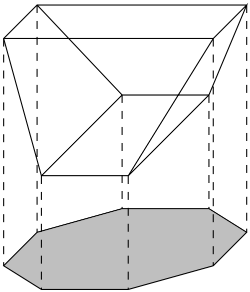
[Ben-Tal and Nemirovski]



Equivalent linear program:

$$\min_{x,y} \ \langle c, x\rangle + \langle 0, y\rangle$$
$$\text{s.t.} \ A_1 x + A_2 y = b,$$
$$x \geq 0, \ y \geq 0$$

Polyhedral lifts of polytopes



lift of P affine slice of $\mathbb{R}^m_+$

$\downarrow \pi$

P polytope

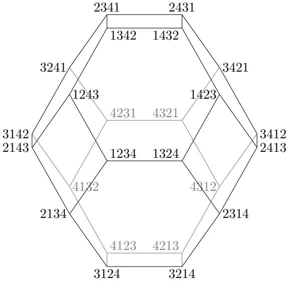
P has polyhedral lift of size $m \quad \longleftrightarrow \quad P = \pi(\mathbb{R}^m_+ \cap L)$

- Key problem: Given polytope P, find smallest lift!

Permutahedron

$$\Pi_n = \text{conv}\{\text{all permutations of } (1, 2, \ldots, n)\}$$

- $n!$ vertices
- $2^n - 2$ facets



Birkhoff polytope

(doubly stochastic matrices)

$$B_n = \left\{ X \in \mathbb{R}^{n \times n} : \begin{array}{r} \sum_i X_{ij} = 1 \ \forall j \\ \sum_j X_{ij} = 1 \ \forall i \\ X_{ij} \geq 0 \ \forall i, j \end{array} \right\}$$

- $\Pi_n = \text{projection}(B_n)$
- Lift of size n^2

Permutahedron

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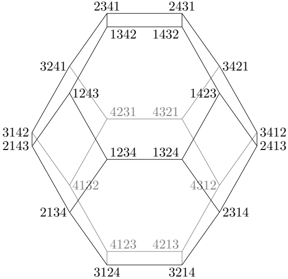

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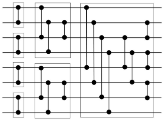
- $\Pi_n = \text{projection}(B_n)$
- Lift of size n^2

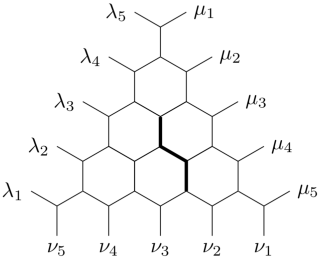
Goemans (2015):
Π_n has a size $O(n \log(n))$ lift



Honeycomb lift of the Horn cone

Horn cone: Horn(n)

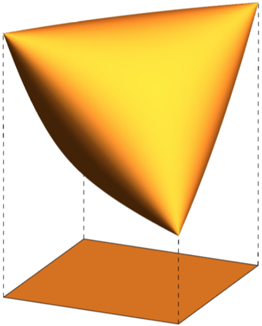
$\{(\lambda(A), \lambda(B), \lambda(C))\ :\ A,\ B,\ C \text{ Hermitian and } A + B + C = 0\}$



Knutson-Tao (1999): Horn(n) has polyhedral lift of size $O(n^2)$

- Allows efficient solution of certain decision problems related to representation theory of $GL(n)$

Conic lifts of convex sets



lift of C — affine slice of a closed convex cone K

$\downarrow \pi$

C — convex set

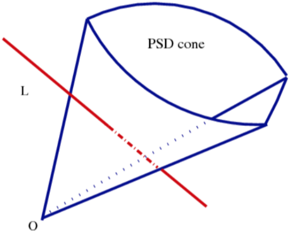
$$C \text{ has a } K\text{-lift} \quad \longleftrightarrow \quad C = \pi(K \cap L)$$

- Key question: Given C and K, does C have a K-lift?

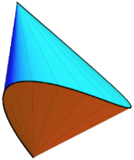
Semidefinite programming

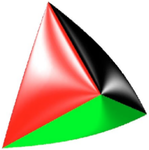
Min. of linear functional over (not too large) spectrahedron.

$$\min_X \ \langle C, X \rangle$$
$$\text{s.t. } \mathcal{A}(X) = b, \quad X \succeq 0$$

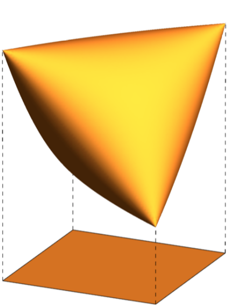


- Spectrahedron affine slice of some positive semidefinite cone
- ... of size m: affine slice of $\mathcal{S}_+^m$





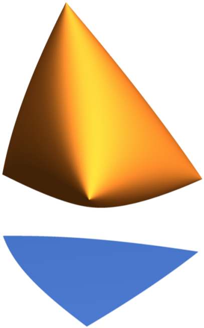
Spectrahedral lifts of convex sets



$\mathcal{S}_+^m$-lift of C

$\downarrow \pi$

C



$$\{\text{Projections of spectrahedra}\} \supsetneq \{\text{spectrahedra}\}$$

- Given C, does it have a spectrahedral lift?
- If so find the smallest spectrahedral lift.

Nonnegativity and sums of squares

Nonnegative polynomials

$$\mathrm{Pol}_+(n, 2d) = \{p \;:\; p(x) \geq 0 \;\; \forall x \in \mathbb{R}^n\}$$

In general, hard to check nonnegativity

Sums of squares: $\mathrm{SOS}(n, 2d) = \{p \;:\; p(x) = \sum_i [q_i(x)]^2\}$

$\mathrm{SOS}(n, 2d)$ has a spectrahedral lift of size $\binom{n+d}{d}$

Hilbert: $\mathrm{SOS}(n, 2d) = \mathrm{Pol}_+(n, 2d)$
$\iff$ $n = 1$, $2d = 2$ or $(n, 2d) = (2, 4)$

Scheiderer (2018): In every other case $\mathrm{Pol}_+(n, 2d)$ has no spectrahedral lift

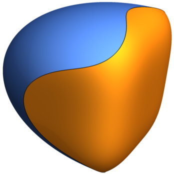
Epigraphs of SOS-convex polynomials

Convex polynomial: first-order characterization

$$D_p(x, y) = p(x) - [p(y) + \langle \nabla p(y), x - y \rangle] \in \mathrm{Pol}_+(2n, 2d)$$

SOS-convex polynomial: $D_p(x, y) \in \mathrm{SOS}(2n, 2d)$

Epigraph:
$\mathrm{epi}(p) = \{(x, t) \; : \; p(x) \leq t\}$



(FGPST 2020): If p is SOS convex then

$$\mathrm{epi}(p) = \{(x, t) \; : \; t - [p(y) + \langle \nabla p(y), x - y \rangle] \in \mathrm{SOS}(n, 2d)\}$$

gives a spectrahedral lift of size $\binom{n+d}{d}$

Slack matrices

Polytope

$$P = \{x \; : \; \langle f_j, x\rangle \leq 1, \; j \in [f]\}$$

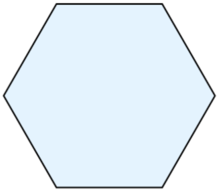
- v vertices
- f facets

Slack matrix

$$[S_P]_{ij} = 1 - \langle f_j, v_i\rangle$$

- $v \times f$ matrix
- entry-wise nonnegative

Example: regular hexagon



$$\begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$

Cone factorization

Dual cone: $K^* = \{y : \langle y, x \rangle \geq 0 \text{ for all } x \in K\}$

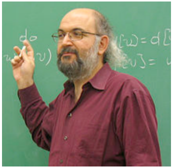
K-factorization of S_P

- a map from vertices to K
- b map from facets to K^*

such that

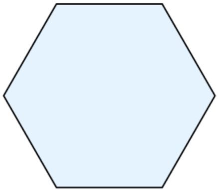
$$[S_P]_{ij} = 1 - \langle f_j, v_i \rangle = \langle b_j, a_i \rangle \ \ \forall i, j$$

Note: $\mathbb{R}^m_+$-factorization is same as non-negative factorization



Yannakakis (1991):

P has $\mathbb{R}^m_+$-lift $\iff$ S_P has $\mathbb{R}^m_+$-factorization



Inequality description:

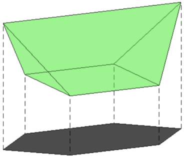
$$\pm x \pm \ y/\sqrt{3} \le 1$$
$$\pm 2y/\sqrt{3} \le 1$$

$\mathbb{R}^5_+$-factorization

$$S_P = \begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Longrightarrow$ Regular hexagon has $\mathbb{R}^5_+$-lift

$$\left\{ y \in \mathbb{R}^5_+ \ : \ \begin{matrix} y_1 + y_2 + y_3 + y_5 = 2 \\ y_3 + y_4 + y_5 = 1 \end{matrix} \right\}$$

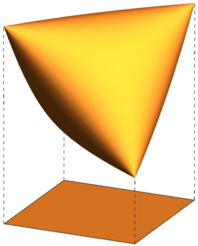


Characterizing K-lifts

Given

- convex body C and
- closed convex cone K

when is $C = \pi(K \cap L)$?



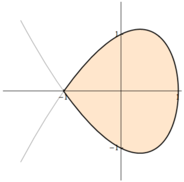
Gouveia, Parrilo, Thomas (2010): If K is 'nice'

$$C \text{ has a } K\text{-lift} \quad \Longleftrightarrow \quad S_C \text{ has a } K\text{-factorization}$$

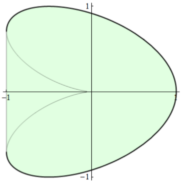
- Generalizes Yannakakis' theorem
- $\rightarrow$ systematic way to find constructions and obstructions

Slack operator: $S_C : \text{ext}(C) \times \text{ext}(C^\circ) \to \mathbb{R}$,
$S_C(x, y) = 1 - \langle y, x \rangle$

ext(C): $[-\sqrt{2}, \sqrt{2}] \ni t \mapsto (1 - t^2, t(2 - t^2))$



ext(C°): $[-\sqrt{2}, \sqrt{2}] \ni s \mapsto \left(\frac{2-3s^2}{s^4+(2-s^2)}, \frac{2s}{s^4+(2-s^2)}\right)$

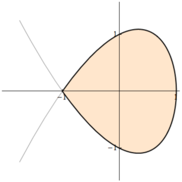


$$S_C(s, t) = \frac{(t - s)^2((2 - t^2) + (s + t)^2)}{s^4 + (2 - s^2)}$$

Slack operator: $S_C : \text{ext}(C) \times \text{ext}(C^\circ) \to \mathbb{R}$,
$$S_C(x, y) = 1 - \langle y, x \rangle$$

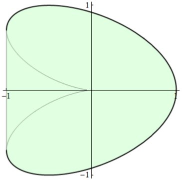
$\mathcal{S}^3_+$-factorization: $\implies$ $\mathcal{S}^3_+$-lift

$$A(t) = \begin{bmatrix} 1 & 0 & 1-t^2 \\ 0 & 2-t^2 & t(2-t^2) \\ 1-t^2 & t(2-t^2) & 1 \end{bmatrix} \quad \forall t \in [-\sqrt{2}, \sqrt{2}]$$



$\mathcal{S}^3_+$-factorization: $\implies$ $\mathcal{S}^3_+$-lift

$$B(s) = \frac{1}{s^4+(2-s^2)} \begin{bmatrix} s^2-1 \\ -s \\ 1 \end{bmatrix} \begin{bmatrix} s^2-1 & -s & 1 \end{bmatrix} \quad \forall s \in [-\sqrt{2}, \sqrt{2}]$$



$$S_C(s, t) = \frac{(t-s)^2((2-t^2)+(s+t)^2)}{s^4+(2-s^2)} = \langle B(s), A(t) \rangle$$

Constructing spectrahedral lifts

$C = \mathrm{conv}(X)$ has $\mathcal{S}_+^m$-lift $\iff$ $\exists$ subspace V of fns on X s.t.

- $\dim(V) \leq m$
- If $\ell(x) \leq 1$ is valid for C then

$$1 - \ell|_X = \sum_k h_k^2 \text{ for } h_k \in V$$

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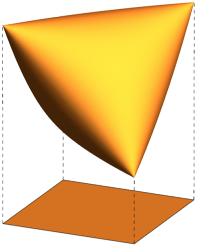


$$X = \{(\pm 1, \pm 1)\} = \{(x,y) : x^2 = y^2 = 1\}$$

$$C = \mathrm{conv}(X) = [-1,1]^2$$

$$V = \mathrm{span}(1, x, y)$$

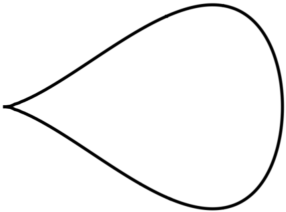
$$1 \pm x = \tfrac{1}{2}(1 \pm x)^2 \;\; \forall (x,y) \in X$$

$$\dim(V) = 3 \quad \Longrightarrow \quad \mathcal{S}_+^3\text{-lift}$$

Which functions to use?

If $C = \text{conv}(X)$ and X is algebraic:

- natural: polynomial functions on X of degree at most d
- Doesn't always work!



Piriform curve:

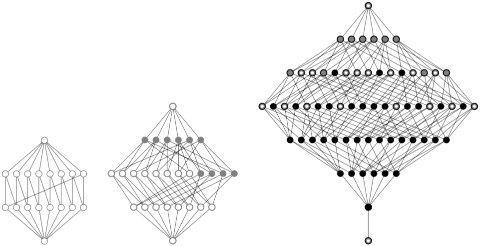
$$y^2 - x^3 + x^4 = 0$$

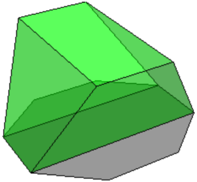
Scheiderer: If X algebraic and has spectrahedral lift, suffices to choose a subspace of semialgebraic functions on X

Obstructions from facial structure

- Obstructions to factorization $\longrightarrow$ obstructions to lifts
- $0-1$ pattern of slack related to facial structure

If $C = \pi(K \cap L)$ then
poset of faces of C embeds into poset of faces of K

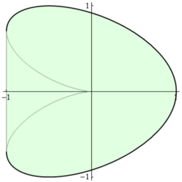




Goemans (2015): P a polytope with v vertices
$\implies$ any $\mathbb{R}^m_+$-lift needs $m \geq \lfloor \log_2(v) \rfloor$

Implies size $O(n \log(n))$ lift of permutahedron is optimal

C has K-lift $\implies$ length of longest chain of faces of P
$<$ length of longest chain of faces of K



- does not have $\mathcal{S}^2_+$-lift
- does not have smooth cone lift

More elaborate obstructions based on neighborliness S. 2020

Algebraic obstructions

If K is semialgebraic then $\pi(K \cap L)$ is semialgebraic

Corollary: C has a spectrahedral lift $\implies$ C semialgebraic

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Algebraic degree of boundary:

- smallest degree of polynomial that vanishes on boundary
- generalizes number of facets of polyhedron
- Algebraic degree of $\partial\mathcal{S}_+^m$ is m (determinant)

Fawzi, El Din (2018):
If $\deg(\partial C) = d$ and C has a $\mathcal{S}_+^m$-lift then $m \geq \sqrt{\log(d)}$

Recent prominent negative results

- Fiorini et al. (2015) traveling salesman polytopes need exponential size polynedral lifts
- Rothvoss (2013) Matching polytope of K_{2n} needs exponential size polyhedral lifts
- Scheiderer (2018) $\mathrm{Pol}_+(2, 6)$ has no spectrahedral lift
- Lee et al. (2015) traveling salesman polytopes need spectrahedral lifts of size $2^{\Omega(n^{1/13})}$

Many open questions!

Some of my favourites...

- Is there a family of polytopes with a big gap between the size of smallest polyhedral and spectrahedral lifts? (Biggest known gap Fawzi, S., Parrilo $\Omega(n/\log(n))$)
- Smallest dimension in which there is a convex semialgebraic set that does not have a spectrahedral lift? (must be ≥ 3)
- Does the matching polytope have a polynomial sized spectrahedral lift?

Thank You!

More information:

Fawzi, Gouveia, Parrilo, Saunderson, Thomas,
"Lifting for simplicity, concise descriptions of convex sets"
`https://arxiv.org/abs/2002.09788`