On Error Bound Moduli for Locally Lipschitz and Regular Functions

(subtitle: far and near end sets)

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Definition

Consider a closed convex set $C \subset \mathbb{R}^n$ and a point $x_0 \in \mathbb{R}^n$. The far end of C relative to x_0 is defined by

$$\text{far}(C, x_0) := \{x \in C \mid x_0 + t(x - x_0) \notin C \; \forall t > 1\},$$

while the near end of C relative to x_0 is defined by

$$\text{near}(C, x_0) := \begin{cases} \{x \in C \mid x_0 + t(x - x_0) \notin C \; \forall t < 1\} & \text{if } x_0 \notin C, \\ \{x_0\} & \text{if } x_0 \in C. \end{cases}$$

See [Hu (2005), Hu (2007)] for the definition of $\text{far}(C, 0)$.

Some basic properties of far and near ends are summarized in the following lemmas.

Lemma

For a closed and convex set $C \subset \mathbb{R}^n$, the following hold:

(a) $\operatorname{far}(C, x_0) = C$ *for all* $x_0 \notin \operatorname{aff} C$.

(b) $\operatorname{far}(C, x_0) \cap \operatorname{ri} C = \emptyset$ *for all* $x_0 \in C$ *or* $x_0 \in \operatorname{aff} C \setminus C$.

(c) $\operatorname{far}(C, x_0) \neq \emptyset$ *if and only if* $\operatorname{pos}(C - x_0) \setminus C^\infty \neq \emptyset$.

Similarly for $\operatorname{near}(C, x_0)$.

Let the shadow of C relative to x_0 be

$$\text{shad}(C, x_0) := \bigcup_{x \in C} \{x_0 + t(x - x_0) \mid t \geq 1\}.$$

Lemma

For a closed and convex set $C \subset \mathbb{R}^n$ and a point $x_0 \notin C$, the following hold:

(a) $\text{far}(C, x_0) = \text{far}\left(\text{cl}(\text{conv}(C \cup \{x_0\})), x_0\right)$.

(b) $\text{near}(C, x_0) = \text{near}\left(\text{shad}(C, x_0), x_0\right)$.

A face of C is a convex subset C' of C such that every closed line segment in C with a relative interior point in C' has both end points in C'.

An exposed face of C is a face of C that is the intersection of C and a supporting hyperplane to C.

The support function $\sigma_C : \mathbb{R}^n \to \overline{\mathbb{R}}$ of C is defined by

$$\sigma_C(x) := \sup_{w \in C} \langle x, w \rangle.$$

$\partial\sigma_C(w) = \arg\max_{v \in C} \langle v, w \rangle = C \cap \{v \in \mathbb{R}^n \mid \langle v, w \rangle = \sigma_C(w)\}.$

F is a nonempty exposed face of C if and only if $F = \partial\sigma_C(w)$ for some $w \neq 0$.

Theorem

Consider a closed and convex set $C \subset \mathbb{R}^n$ and a point $x_0 \in \mathbb{R}^n$ with $\underline{x_0 \in C}$. Then the following hold:

(a) $\text{far}(C, x_0)$ *consists of all the faces F of C such that $x_0 \notin F$.*

(b) *The set consisting of all the exposed faces F of C such that $x_0 \notin F$, is a dense subset of* $\text{far}(C, x_0)$, *or in other words,*

$$\bigcup_{\sigma_C(w) > \langle x_0, w\rangle} \partial\sigma_C(w) \subset \text{far}(C, x_0) \subset \text{cl}\left(\bigcup_{\sigma_C(w) > \langle x_0, w\rangle} \partial\sigma_C(w)\right).$$

Similarly for $x_0 \notin C$.

Theorem

For a closed and convex set $C \subset \mathbb{R}^n$ and a point $\underline{x_0 \notin C}$, the following hold in terms of $C' := \operatorname{shad}(C, x_0)$:

(a) $\operatorname{near}(C, x_0)$ *consists of all the common faces F of C and C' such that $x_0 \notin \operatorname{aff} F$.*

(b) *The set consisting of all the common exposed faces F of C and C' such that $x_0 \notin \operatorname{aff} F$, is a dense subset of* $\operatorname{near}(C, x_0)$*, or in other words,*

$$\bigcup_{\sigma_C(w) < \langle x_0, w \rangle} \partial \sigma_C(w) \subset \operatorname{near}(C, x_0) \subset \operatorname{cl}\left(\bigcup_{\sigma_C(w) < \langle x_0, w \rangle} \partial \sigma_C(w) \right).$$

[Hu (2005)] showed that for a convex inequality $f(x) \leq 0$,

$$\text{strong BCQ} = \text{ BCQ } + d(0, \text{far}(\partial f(x), 0)) > 0.$$

[Hu (2007)] studied the global error bounds for the level set $S := [f \leq 0]$ by a weak BCQ and $d(0, \text{far}(\partial f(x) \cap N_S(x), 0)) > 0$ in a Banach space.

[Meng, Roshchina and Y. (2015)] studied the exact tangent approximation of C and relative continuity of the gauge function, by the equivalence of global error bound of the support function and $d(0, \mathrm{far}(C, 0)) > 0$.

[Zheng and Ng (2004)] implicitly used $d(0, \mathrm{far}(\partial f(x), 0)) > 0$ to show that the metric subregularity of the solution set of a generalized equation is equivalent to strong BCQ.

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For a closed and convex set C with $0 \in C$, the gauge of C is the function $\gamma_C : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$\gamma_C(x) := \inf\{\lambda \geq 0 | x \in \lambda C\},$$

which is lower semicontinuous and sublinear with $\mathrm{dom}(\gamma_C) = \mathrm{pos}(C)$.

For a closed and convex set $C \subset \mathbb{R}^n$ with $0 \notin C$, the co-gauge of C is the function $\nu_C : \mathbb{R}^n \to \overline{\mathbb{R}}$ defined by

$$\nu_C(x) := \sup\{\lambda \geq 0 \mid x \in \lambda C\},$$

which is upper semicontinuous and suplinear with $\mathrm{dom}(-\nu_C) = \mathrm{cl}(\mathrm{pos}\, C)$.

If a sublinear function $p : \mathbb{R}^n \to \mathbb{R}$ is continuous at 0, then it is continuous everywhere.

For a sublinear function not defined on the whole space:

$$p(x,y) = \begin{cases} |y|, & x = 0 \\ 0, & x > 0 \end{cases}$$

it is continuous at $(0,0)$ relative to $\{(x,y)|x \geq 0\}$, but not continuous relative to $\{(x,y)|x \geq 0\}$ anywhere else on $x = 0$.

Theorem

For a closed and convex set $C \subset \mathbb{R}^n$ with $0 \in C$, the following are equivalent:

(a) γ_C *is continuous at 0 relative to* $\operatorname{pos} C$ *if and only if* $d(0, \operatorname{far}(C, 0)) > 0$;

(b) γ_C *is continuous relative to its domain* $\operatorname{pos} C$ *if and only if* $\operatorname{far}(C, 0)$ *is closed.*

Theorem

For a closed and convex set $C \subset \mathbb{R}^n$ *such that* $0 \notin C$ *and* $\lambda C \subset C$ *for all* $\lambda \geq 1$, *we have*

$$\mathrm{dom}(-\nu_C) = C^\infty,$$

and the following hold:

(a) ν_C *is continuous at every* $x \in \{0\} \cup (\mathrm{ri}\, C^\infty) \cup (C^\infty \setminus \mathrm{pos}\, C)$ *relative to* C^∞.

(b) ν_C *is continuous relative to* C^∞ *if and only if* $\mathrm{near}(C, 0)$ *is closed and*

$$(\mathrm{near}(C, 0))^\infty \cap \mathrm{pos}\, C = \{0\}.$$

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Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be lsc and $\bar{x} \in [f \leq 0]$ (the level set).

Definition

f has a local error bound for $[f \leq 0]$ at $\bar{x}$, if there exist some $\tau > 0$ and $\epsilon > 0$ such that,

$$\tau d(x, [f \leq 0]) \leq f(x), \quad \text{with } \|x - \bar{x}\| < \epsilon. \tag{1}$$

f has a global error bound for $[f \leq 0]$ at $\bar{x}$ if the above inequality holds for all x.

The **local error bound modulus** of f at $\bar{x}$ is defined by

$$\text{ebm}(f, \bar{x}) := \liminf_{x \to \bar{x}, f(x) > 0} \frac{f(x)}{d(x, [f \leq 0])}.$$

Clearly, $0 \leq \text{ebm}(f, \bar{x}) \leq +\infty$.

As $\mathrm{ebm}(f, \bar{x}) = +\infty$ whenever $\bar{x} \in \mathrm{int}([f \leq 0])$, we assume in what follows that $\bar{x} \in \mathrm{bdry}[f \leq 0]$.

The outer limiting subdifferential of f at $\bar{x}$ is defined by

$$\partial^{>} f(\bar{x}) := \left\{ \lim_{k\to+\infty} v_k \mid \exists x_k \to_f \bar{x},\ f(x_k) > f(\bar{x}),\ v_k \in \partial f(x_k) \right\}$$

A lower estimate via outer limiting subdifferential:

If $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is lower semicontinuous, we have

$$d(0, \partial^{>} f(\bar{x})) \leq \mathrm{ebm}(f, \bar{x}),$$

If $0 < d(0, \partial^{>} f(\bar{x}))$, then f has a local error bound at $\bar{x}$.

Convex and lower semicontinuous case

If f is convex lower semicontinuous, we have

$$d(0, \partial^{>} f(\bar{x})) = \mathrm{ebm}(f, \bar{x}).$$

See [Kruger et al (2010), Fabian et al (2010), Ioffe (2015)].

Sublinear case

It was shown by [Hu and Wang (2011)] that, if f is a sublinear and lower semicontinuous function, then

$$\operatorname{ebm}(f,0) = d(0, \operatorname{far}(C,0)),$$

where C is the unique closed and convex set such that $f = \sigma_C$.

In this case, we also have

$$S \subset \operatorname{far}(C,0) \subset \partial^{>}\sigma_C(0) \subset \operatorname{cl} S.$$

Thus

$$\operatorname{ebm}(f,0) = d(0, \partial^{>} f(0)) = d(0, \partial^{>}\sigma_C(0)).$$

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ *and* $\bar{x} \in \mathrm{bdry}[f \leq 0]$*. If* f *is* **locally Lipschitz and regular** *at* $\bar{x}$*, then*

$$d\left(0, \partial^{>} f(\bar{x})\right) \leq \mathrm{ebm}(f, \bar{x}) \leq d(0, \partial^{>} \sigma_{\partial f(\bar{x})}(0)) \equiv d(0, \mathrm{far}(\partial f(\bar{x}), 0)).$$

Let f be lower-$\mathcal{C}^1$:

$$f(x) = \max_{y \in Y} \phi(x, y)$$

in which each function $\phi(\cdot, y)$ is of class $\mathcal{C}^1$ and the index set $Y \subset \mathbb{R}^m$ is compact.

Let the active index set mapping $Y : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be defined by

$$Y(x) := \{y \in Y | \phi(x, y) = f(x)\}.$$

We introduce two collections of index sets as follows:

$$\mathcal{Y}(\bar{x}) := \left\{ \lim_{k \to +\infty} Y(x_k) \mid x_k \to \bar{x} \text{ and } f(x_k) > 0 \ \forall k \right\},$$

(outer limiting active index set.)

$$\mathcal{Y}^{>}(\bar{x}) := \left\{ \arg\max_{y \in Y(\bar{x})} \langle \nabla_x \phi(\bar{x}, y), w \rangle \middle| \exists w : \max_{y \in Y(\bar{x})} \langle \nabla_x \phi(\bar{x}, y), w \rangle > 0 \right\}$$

(optimal active index set.)

Theorem

If f is lower-$\mathcal{C}^1$, then

$$d(0, \partial^{>} f(\bar{x})) = \mathrm{ebm}(f, \bar{x}) \leq d(0, \mathrm{far}(\partial f(\bar{x}), 0)).$$

Furthermore, we have

$$\partial^{>} f(\bar{x}) = \bigcup_{Y' \in \mathcal{Y}(\bar{x})} \mathrm{conv}\{\nabla_x \phi(\bar{x}, y) | y \in Y'\},$$

and

$$\bigcup_{Y' \in \mathcal{Y}^{>}(\bar{x})} \mathrm{conv}\{\nabla_x \phi(\bar{x}, y) | y \in Y'\} \subset \mathrm{far}(\partial f(\bar{x}), 0)$$

$$\subset \mathrm{cl} \bigcup_{Y' \in \mathcal{Y}^{>}(\bar{x})} \mathrm{conv}\{\nabla_x \phi(\bar{x}, y) | y \in Y'\}.$$

sharp upper bound for convex functions.

Theorem

Assume that f is finite and convex on some convex neighborhood of $\bar{x}$. If the Abadie's CQ holds, i.e., $[df(\bar{x}) \leq 0] = T_{[f\leq 0]}(\bar{x})$, and the level set $[f \leq 0]$ admits exact tangent approximation, i.e., there is a neighborhood V of $\bar{x}$ such that

$$[f \leq 0] \cap V = (\bar{x} + T_{[f\leq 0]}(\bar{x})) \cap V,$$

then the following equalities hold:

$$d\left(0, \partial^{>} f(\bar{x})\right) = \mathrm{ebm}(f, \bar{x}) = d\left(0, \partial^{>} \sigma_{\partial f(\bar{x})}(0)\right).$$

Consider the linear semi-inf system

$$\langle a_y, x\rangle \le b_y \quad \forall y \in Y,$$

where Y is a compact space and $a_y \in \mathbb{R}^n$ and $b_y \in \mathbb{R}$ depend continuously on $y \in Y$.

- $f(x) := \max_{y\in Y}\{\langle a_y, x\rangle - b_y\}$;
- $Y(x) := \{y \in Y \mid \langle a_y, x\rangle - b_y = f(x)\}$.

According to [Anderson and Goberna (1998)], the linear semi-inf system is said to be a locally polyhedral if

$$(\operatorname{pos}\operatorname{conv}\{a_y \mid y \in Y(x)\})^* = \operatorname{pos}([f \le 0] - x) \quad \forall x \in [f \le 0].$$

Corollary

If one of the following equivalent properties is satisfied:

(a) *The exact tangent approximation condition holds at $\bar{x}$,*

(b) *The linear semi-inf system is locally polyhedral,*

then,

$$\mathrm{ebm}(f,x) = d\left(0, \bigcup_{Y'\in\mathcal{Y}(x)} \mathrm{conv}\{a_y \mid y \in Y'\}\right),$$

$$\mathcal{Y}(x) := \{Y' | \exists w : \langle a_y, w\rangle = 1\, \forall y \in Y', \langle a_y, w\rangle < 1\, \forall y \in Y(x)\backslash Y'\}.$$

Remark: As a finite linear system is naturally locally polyhedral, our result recovers [Cánovas et al (2014), Theorem 4.1] for the case of a finite linear system.

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• The far$(\partial f(x), 0)$, in particular $d(0, \text{far}(\partial f(x), 0) > 0$, plays an important role of a regularity assumption.

• The modulus of weak sharp minumum has been used in convergence analysis for some optimal algorithms. We anticipate that the upper bound of the local error bound modulus will play a similar role.

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