Constraint Splitting and Projection Methods for Optimal Control

Variational Analysis and Optimization Webinar
Mathematics of Computation and Optimization (MoCaO)
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Early part based on

- H. H. Bauschke, R. S. Burachik, C. Y. Kaya, Constraint splitting and projection methods for optimal control of double integrator. Chapter in Springer Edited Book "Splitting Algorithms, Modern Operator Theory, and Applications," pp. 45–68, 2019. (arXiv:1804.03767, 2018).

Outline

1 Cubic curves in $\mathbb{R}^n$: *Minimum-energy control of the double integrator*

2 Constraint splitting and projections

3 Best approx. algorithms: *Dykstra* | *MAP* | *DR* | *AAC* | *FISTA*

4 Numerical experiments: parametric behaviour | error analysis

Motivation

- Douglas–Rachford splitting method applied to *discrete-time* optimal control problems.
  (O'Donoghue–Stathopoulos–Boyd 2013)
  Also see (Eckstein–Ferris 1998).

- No known example of application of best approximation algorithms to *continuous-time* optimal control problems – except Bauschke–Burachik–K (2019).

- Minimum-energy control of the double integrator – building block for cubic splines.

Cubic Curves in $\mathbb{R}^n$

$$
(\mathrm{P}) \begin{cases} \min & \dfrac{1}{2}\displaystyle\int_0^1 \|u(t)\|_2^2\, dt \\ \text{subject to} & \dot{x}_1(t) = x_2(t)\,, \quad x_1(0) = s_0\,, \quad x_1(1) = s_f\,, \\ & \dot{x}_2(t) = u(t)\,, \quad x_2(0) = v_0\,, \quad x_2(1) = v_f\,. \end{cases}
$$

$x_1(t), x_2(t) \in \mathbb{R}^n$: state variable vectors

$u(t) \in \mathbb{R}^n$: control variable vector

$x_i(t) = (x_{i,1}(t), \ldots, x_{i,n}(t)),\ i = 1, 2; \quad u(t) = (u_1(t), \ldots, u_n(t))$

$n = 1$: Min.-energy control of the the double integrator

$n = 2, 3$: Spatial curves with minimum ave. acceleration

Cubic Curves in $\mathbb{R}^n$

$$
(\mathrm{P}) \begin{cases} \min & \dfrac{1}{2}\displaystyle\int_0^1 \|u(t)\|_2^2\, dt \\ \text{subject to} & \dot{x}_1(t) = x_2(t)\,, \quad x_1(0) = s_0\,, \quad x_1(1) = s_f\,, \\ & \dot{x}_2(t) = u(t)\,, \quad x_2(0) = v_0\,, \quad x_2(1) = v_f\,, \\ & \|u(t)\|_2 \le a\,. \quad \text{(constrained acceleration)} \end{cases}
$$

$x_1(t), x_2(t) \in \mathbb{R}^n$: state variable vectors

$u(t) \in \mathbb{R}^n$: control variable vector

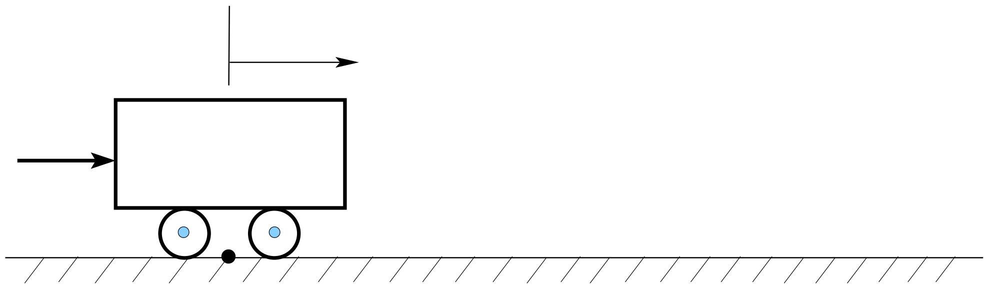
$x_i(t) = (x_{i,1}(t), \ldots, x_{i,n}(t)),\ i = 1, 2; \quad u(t) = (u_1(t), \ldots, u_n(t))$

$n = 1$: Min.-energy control of the the double integrator

$n = 2, 3$: Spatial curves with minimum ave. acceleration

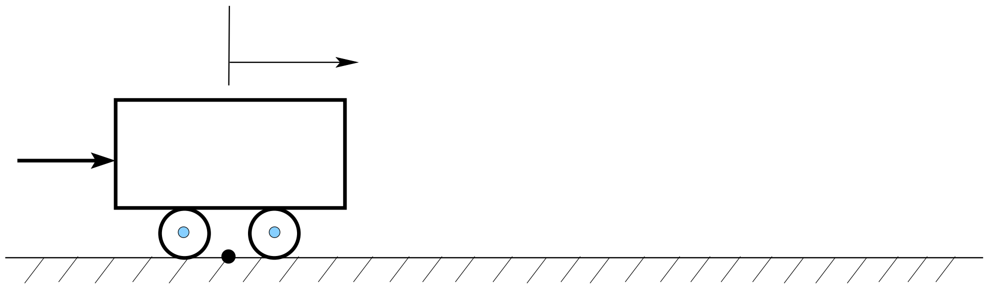
Examples with $n = 1, 2, 3$

$n = 1:$

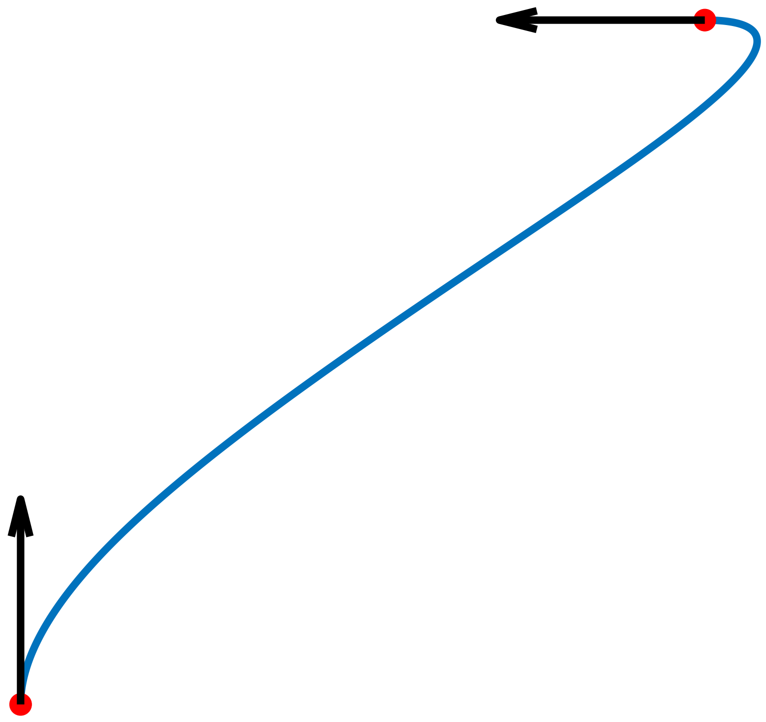


Examples with $n = 1, 2, 3$

$n = 1:$

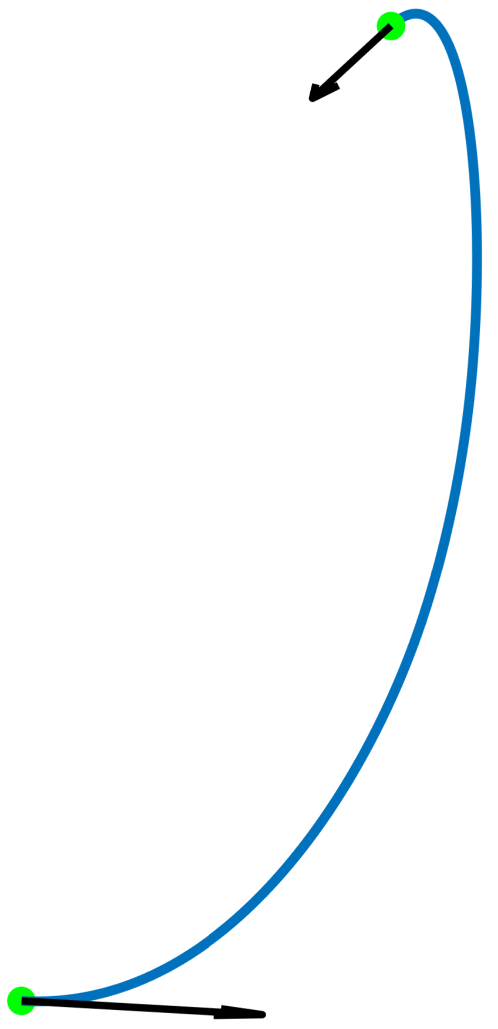


$n = 2:$

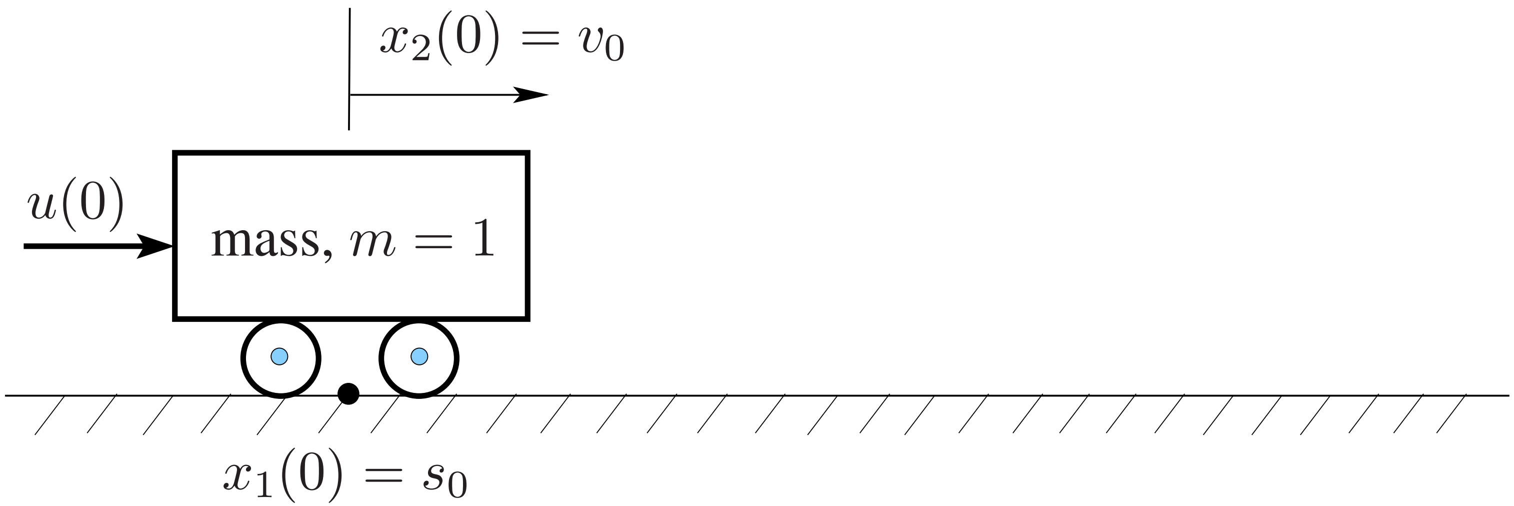


Examples with $n = 1, 2, 3$

$n = 3$:



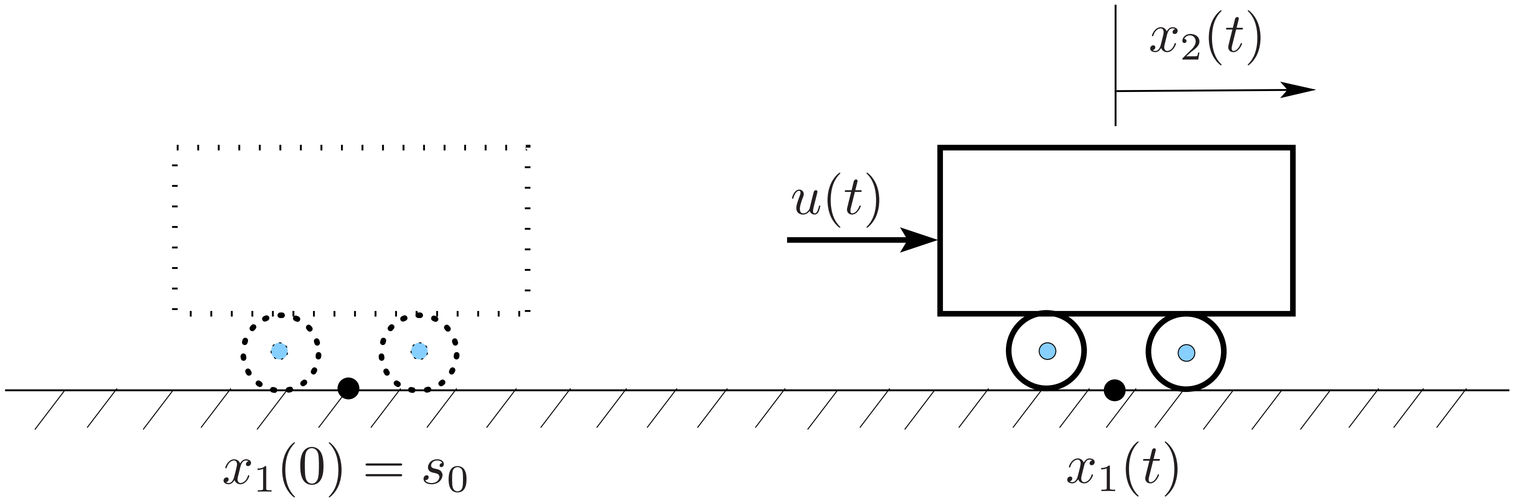
Min.-energy Control of Double Integrator



$$
\text{(P0)} \quad \begin{cases} \min & \displaystyle\frac{1}{2}\int_0^1 u^2(t)\,dt \\ \text{subject to} & \dot{x}_1(t) = x_2(t)\,, \quad x_1(0) = s_0\,, \quad x_1(1) = s_f\,, \\ & \dot{x}_2(t) = u(t)\,, \quad x_2(0) = v_0\,, \quad x_2(1) = v_f\,. \end{cases}
$$

$x_1(t)$, $x_2(t)$: state variables; $u(t)$: control variable.

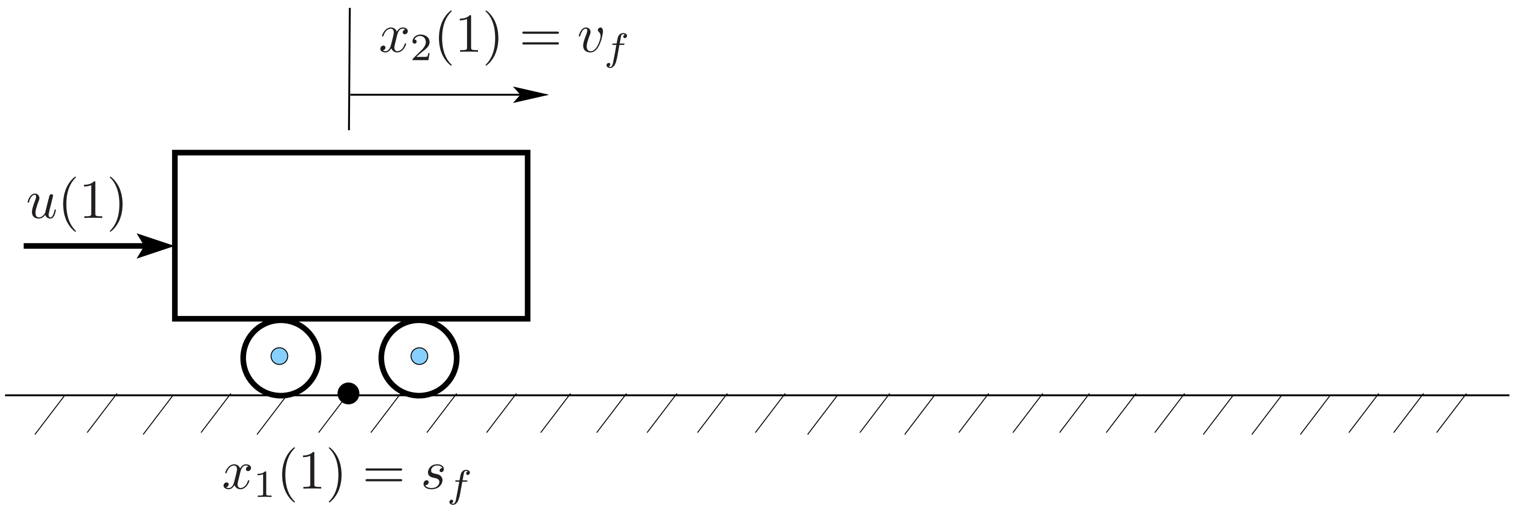
Min.-energy Control of Double Integrator



$$
\text{(P0)} \begin{cases} \min & \dfrac{1}{2}\displaystyle\int_0^1 u^2(t)\,dt \\ \text{subject to} & \dot{x}_1(t) = x_2(t)\,, \quad x_1(0) = s_0\,, \quad x_1(1) = s_f\,, \\ & \dot{x}_2(t) = u(t)\,, \quad x_2(0) = v_0\,, \quad x_2(1) = v_f\,. \end{cases}
$$

$x_1(t)$, $x_2(t)$: state variables; $u(t)$: control variable.

Min.-energy Control of Double Integrator



$$
(\mathrm{P0}) \begin{cases} \min & \dfrac{1}{2}\displaystyle\int_0^1 u^2(t)\,dt \\ \text{subject to} & \dot{x}_1(t) = x_2(t)\,, \quad x_1(0) = s_0\,, \quad x_1(1) = s_f\,, \\ & \dot{x}_2(t) = u(t)\,, \quad x_2(0) = v_0\,, \quad x_2(1) = v_f\,. \end{cases}
$$

$x_1(t)$, $x_2(t)$: state variables; $u(t)$: control variable.

Min.-energy Control of Double Integrator

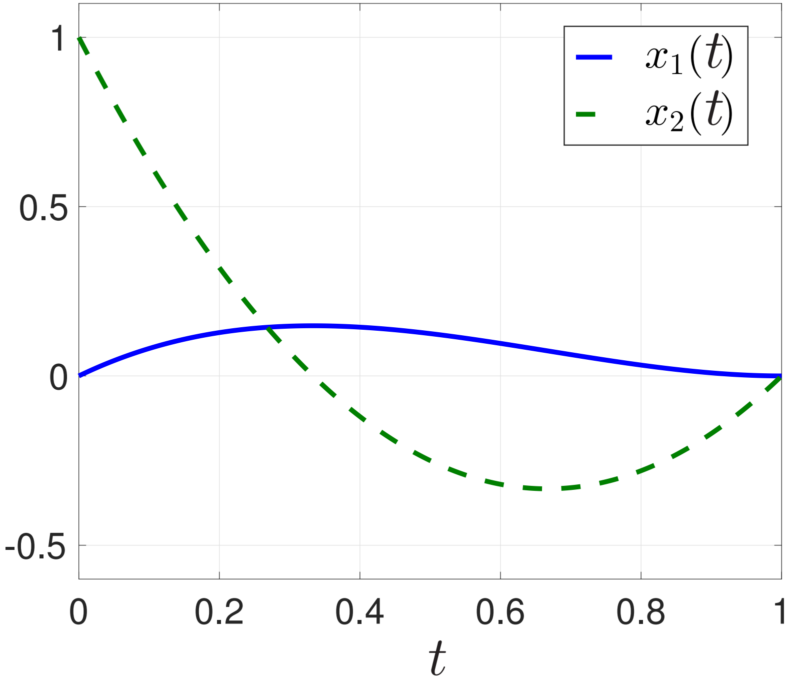
$$\begin{aligned} u(t) &= c_1\, t + c_2\,, \\ x_1(t) &= \frac{1}{6}\, c_1\, t^3 + \frac{1}{2}\, c_2\, t^2 + v_0\, t + s_0\,, \\ x_2(t) &= \frac{1}{2}\, c_1\, t^2 + c_2\, t + v_0\,, \end{aligned}$$

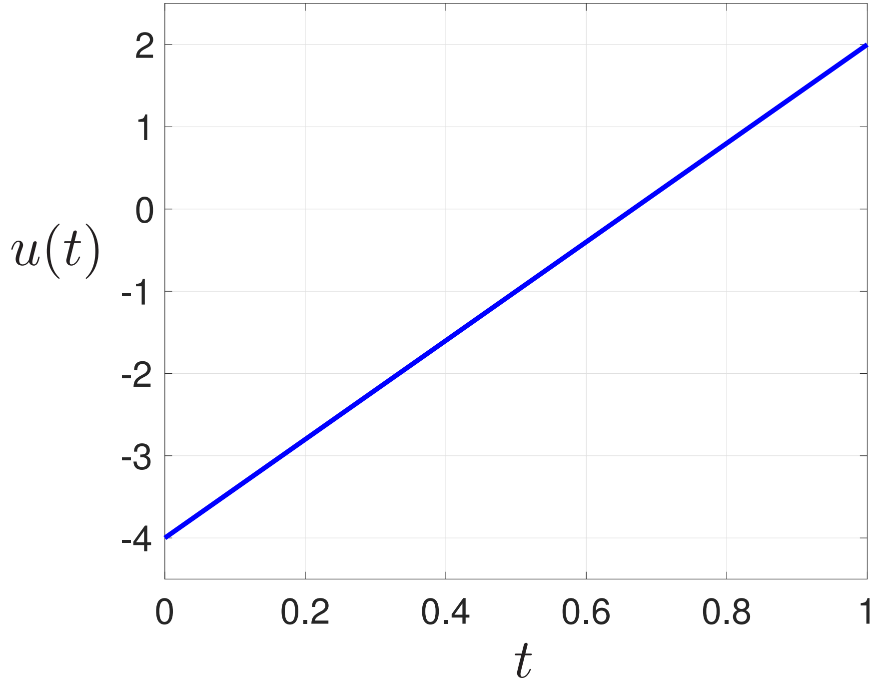
for all $t \in [0, 1]$, where

$$\begin{aligned} c_1 &= -12\,(s_f - s_0) + 6\,(v_0 + v_f)\,, \\ c_2 &= 6\,(s_f - s_0) - 2\,(2\,v_0 + v_f)\,. \end{aligned}$$

Min.-energy Control of Double Integrator

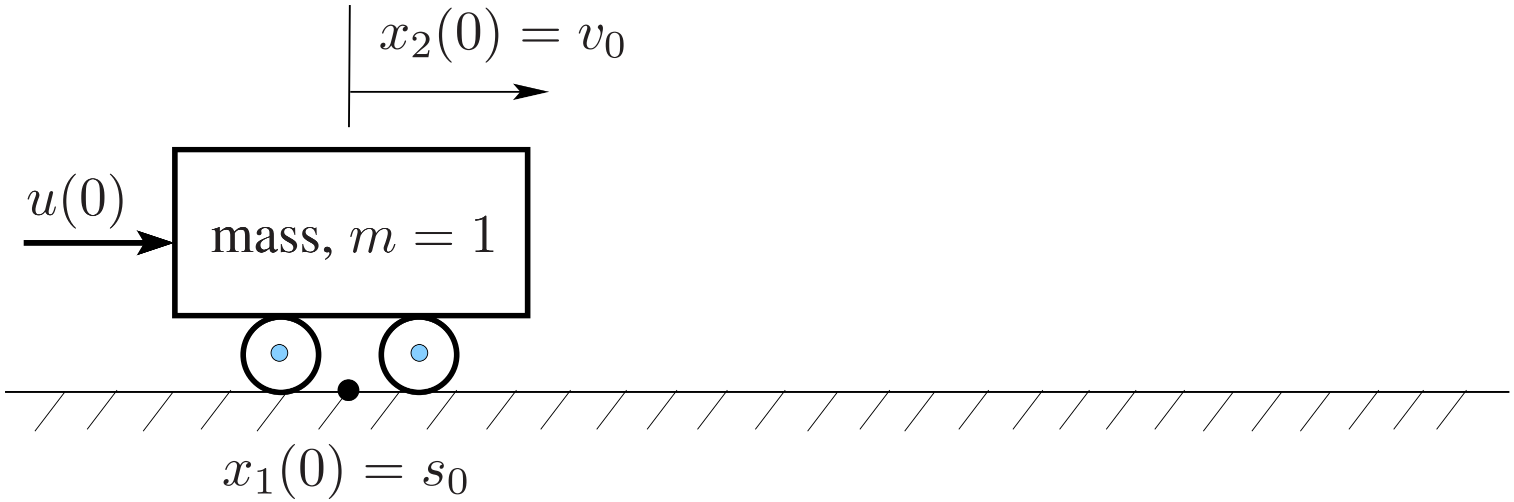
Solution with $s_0 = 0$, $s_f = 0$, $v_0 = 1$, $v_f = 0$:





$$
\begin{aligned}
u(t) &= 6\,t - 4; \\
x_1(t) &= t^3 - 2\,t^2 + t \\
x_2(t) &= 3\,t^2 - 4\,t + 1
\end{aligned}
$$

Min.-energy Control with Constraints



$$
(\mathrm{P}) \begin{cases} \min & \dfrac{1}{2}\displaystyle\int_0^1 u^2(t)\,dt \\ \text{subject to} & \dot{x}_1(t) = x_2(t)\,, \quad x_1(0) = s_0\,, \quad x_1(1) = s_f\,, \\ & \dot{x}_2(t) = u(t)\,, \quad x_2(0) = v_0\,, \quad x_2(1) = v_f\,, \\ & -a \le u(t) \le a\,. \end{cases}
$$

$a > 0$ some real constant.

Min.-energy Control with Constraints

Maximum Principle

Define the *Hamiltonian function* :

$$H(x_1, x_2, u, \lambda_1, \lambda_2) := \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u\,,$$

where the *adjoint variables* $\lambda_1(t)$ and $\lambda_2(t)$ satisfy

$$\dot{\lambda}_1 = -\partial H/\partial x_1 \quad \text{and} \quad \dot{\lambda}_2 = -\partial H/\partial x_2\,,$$

i.e.,

$$\lambda_1(t) = c_1 \quad \text{and} \quad \lambda_2(t) = -c_1\,t - c_2\,,$$

c_1, c_2 real constants.

Optimality Conditions (DI)

Maximum Principle

If u is an optimal control for Problem (P), then there exists a continuously differentiable vector of adjoint variables λ, as defined before, such that $\lambda(t) \neq 0$ for all $t \in [0, t_f]$, and that, for a.e. $t \in [0, t_f]$,

$$u(t) = \operatorname*{argmin}_{v \in [-a,a]} H(x, v, \lambda(t)) \,,$$

i.e.,

$$u(t) = \operatorname*{argmin}_{v \in [-a,a]} \frac{1}{2} v^2 + \lambda_2(t)\, v \,.$$

Optimality Conditions (DI)

Optimal control

$$u(t) = \begin{cases} -\lambda_2(t)\,, & \text{if } -a \le \lambda_2(t) \le a\,, \\ a\,, & \text{if } \lambda_2(t) \le -a\,, \\ -a\,, & \text{if } \lambda_2(t) \ge a\,. \end{cases}$$

Note that the optimal control u for Problem (P) is continuous.

Numerical Solution Techniques

Three approaches:

I. (*First-*)*discretize-then-optimize*

1. Discretize Problem (P) over a partition of the time horizon $[0, 1]$.
2. Use some (large-scale) finite-dimensional optimization software (e.g. AMPL + Ipopt) to get a *discrete* (finite-dimensional) *approximation* for the state and control variables $x(t)$ and $u(t)$.

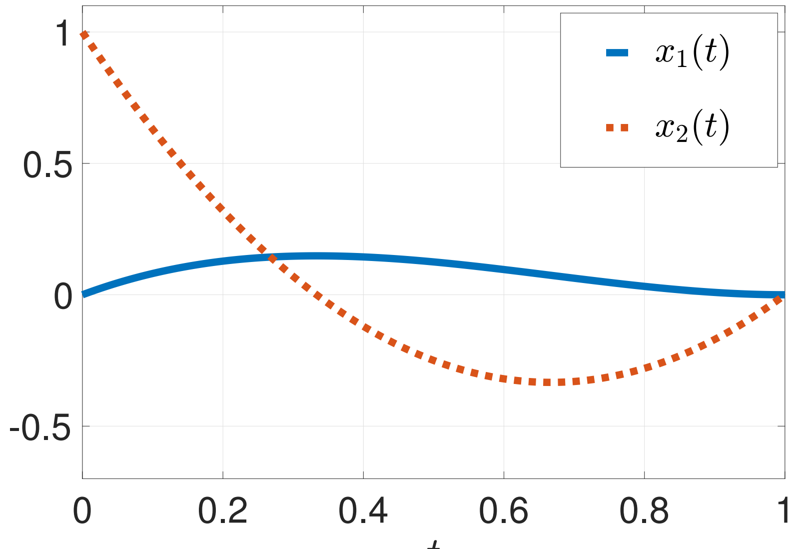
II. (*First-*)*optimize-then-discretize*

1. Write down conditions of optimality.
2. Solve the optimality conditions by using discretized functions.

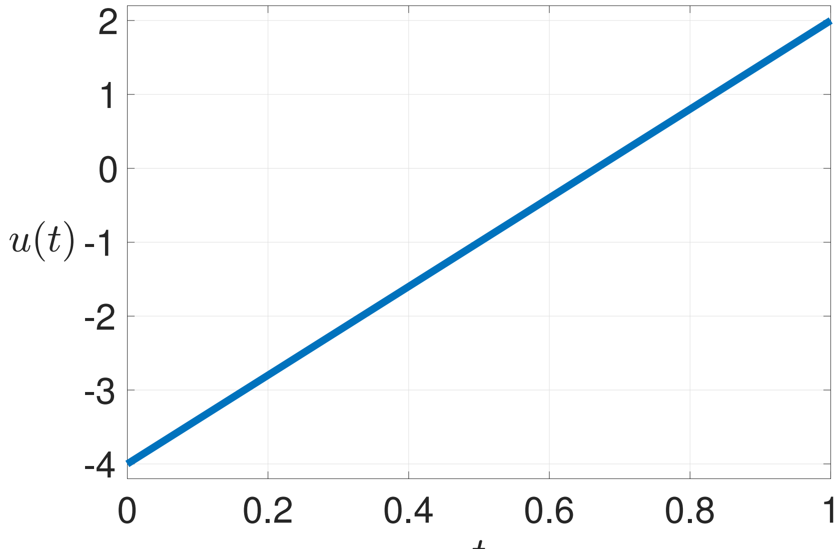
III. *Arc parameterization*

1. Parameterize w.r.t. a concatenation of $(u(t) = a)$-, $(u(t) = -a)$- and $(u(t) = -\lambda_2(t))$-arcs over intervals $[t_{i-1}, t_i]$, t_i unknown, $i = 1, \ldots, N$.
2. Use some finite-dimensional optimization software (e.g. AMPL + Ipopt) to find the unknown t_i, $i = 1, \ldots, N$.

Analytical Solution (DI)



(a) Optimal state variables

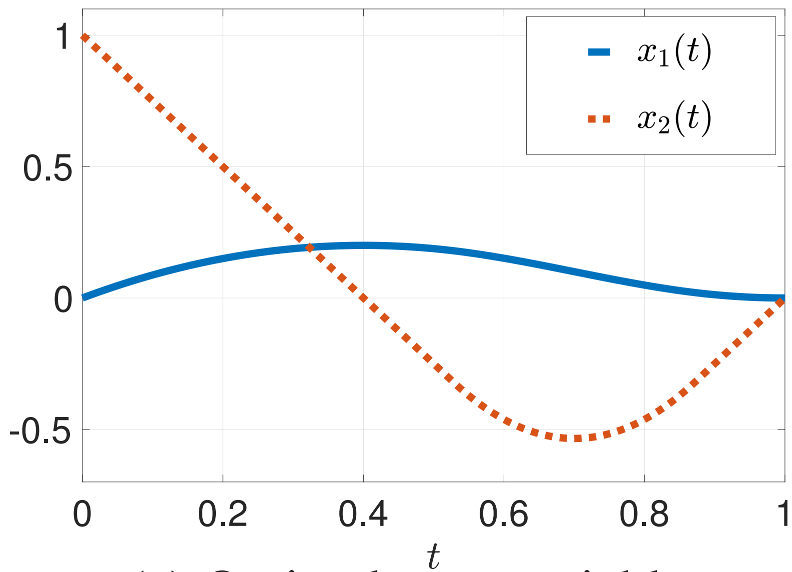


(b) Optimal control variable

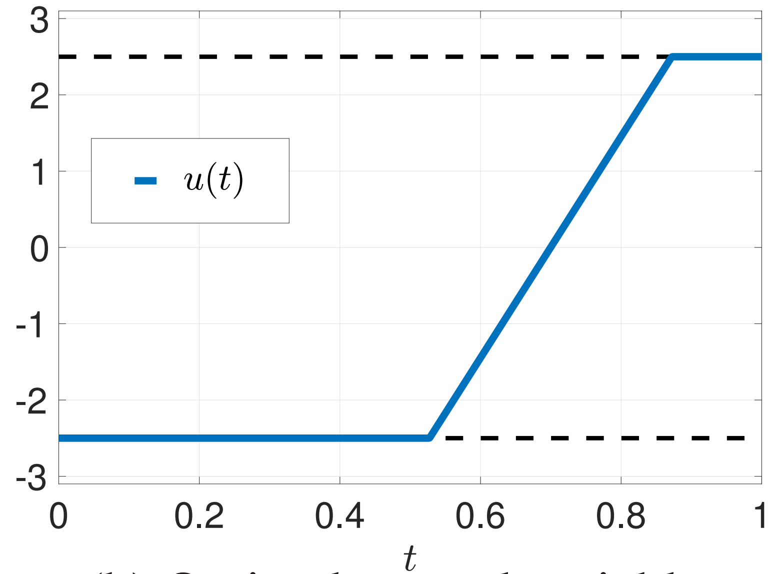
$s_0 = 0$, $s_f = 0$, $v_0 = 1$, $v_f = 0$

$a = \infty$

Numerical Solution (DI)

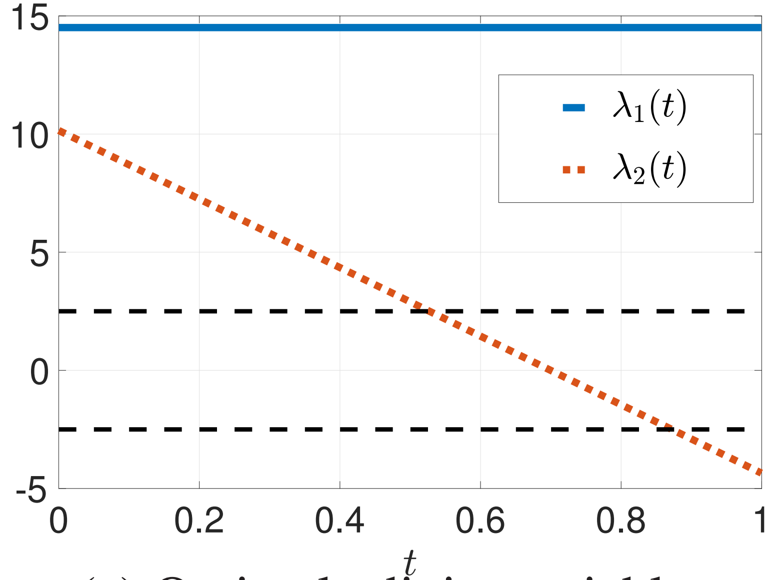


(a) Optimal state variables



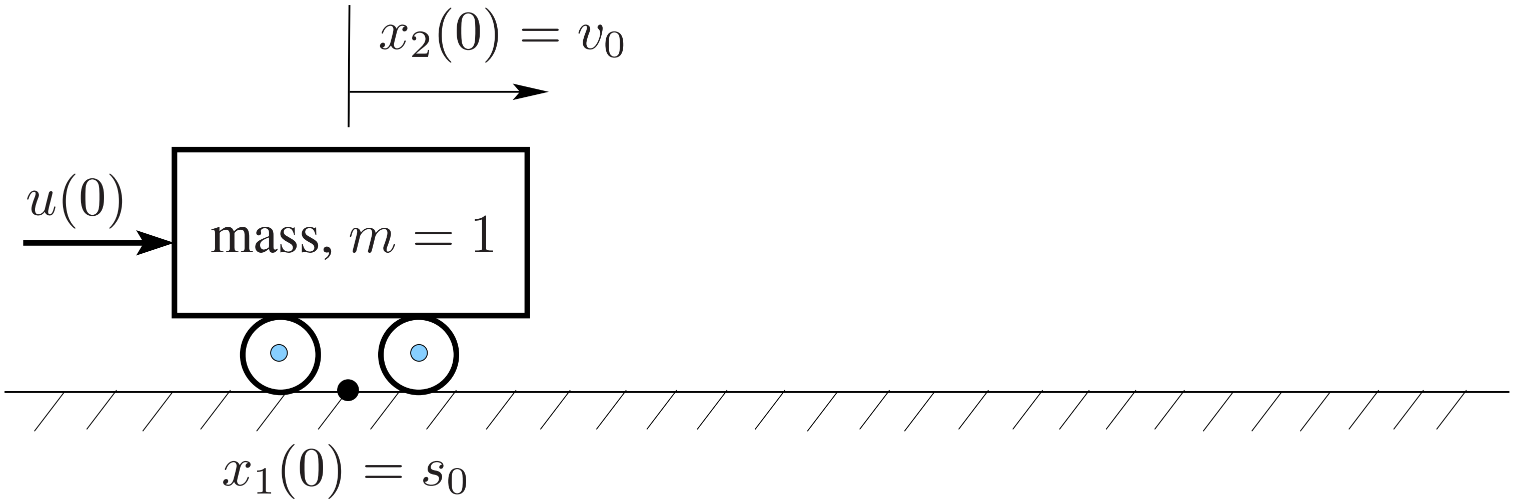
(b) Optimal control variable

$$s_0 = 0,\, s_f = 0,\, v_0 = 1,\, v_f = 0$$
$$a = 2.5$$



(c) Optimal adjoint variables

Constraint Splitting (DI)



$$
(\mathrm{P}) \left\{ \begin{array}{ll}
\min & \displaystyle \frac{1}{2}\int_0^1 u^2(t)\,dt \\
\text{subject to} & \dot{x}_1(t) = x_2(t)\,, \quad x_1(0) = s_0\,, \quad x_1(1) = s_f\,, \\
& \dot{x}_2(t) = u(t)\,, \quad x_2(0) = v_0\,, \quad x_2(1) = v_f\,, \\
& -a \le u(t) \le a\,.
\end{array} \right.
$$

Constraint Splitting (DI in $\mathbb{R}^n$)

$$
\text{(Pc)}\quad \begin{cases}
\quad\text{min} & \displaystyle\frac{1}{2}\int_0^1 \|u(t)\|_2^2\, dt \\
\text{subject to} & \dot{x}_1(t) = x_2(t)\,,\ \ x_1(0) = s_0\,,\ \ x_1(1) = s_f\,, \\
 & \dot{x}_2(t) = u(t)\,,\ \ \ x_2(0) = v_0\,,\ \ x_2(1) = v_f\,, \\
 & \|u(t)\|_2 \le a\,.
\end{cases}
$$

Constraint Splitting

$$\mathcal{A} := \{u \in L^2(0,1;\mathbb{R}^n) \mid \exists x_i \in W^{1,2}(0,1;\mathbb{R}^n), i = 1,2, \text{ which solve}$$

$$\dot{x}_1(t) = x_2(t)\,,\; x_1(0) = s_0\,,\; x_1(1) = s_f\,,$$

$$\dot{x}_2(t) = u(t)\,,\;\; x_2(0) = v_0\,,\; x_2(1) = v_f\,,$$

$$\forall t \in [0,1]\}\,,$$

$$\mathcal{B} := \{u \in L^2(0,1;\mathbb{R}^n) \mid \|u(t)\|_2 \le a\,, \text{ for all } t \in [0,1]\}\,.$$

$\mathcal{A}$ is an ***affine subspace*** and $\mathcal{B}$ a ***ball***.

Projections

The projection onto $\mathcal{A}$ from a current iterate u^- is u which solves

$$\text{(P1)} \begin{cases} \min & \dfrac{1}{2}\displaystyle\int_0^1 \|u(t) - u^-(t)\|_2^2 \, dt \\ \text{subject to} & u \in \mathcal{A}\,. \end{cases}$$

The projection onto $\mathcal{B}$ from a current iterate u^- is u which solves

$$\text{(P2)} \begin{cases} \min & \dfrac{1}{2}\displaystyle\int_0^1 \|u(t) - u^-(t)\|_2^2 \, dt \\ \text{subject to} & u \in \mathcal{B}\,. \end{cases}$$

Projections

Proposition 1 (Projection onto $\boldsymbol{\mathcal{A}}$). *The projection $P_{\mathcal{A}}$ of $u^- \in L^2(0,1;\mathbb{R}^n)$ onto the constraint set $\mathcal{A}$, as the solution of Problem* (P1)*, is given by*

$$P_{\mathcal{A}}(u^-)(t) = u^-(t) + c_1\, t + c_2\,,$$

for all $t \in [0,1]$, where

$$c_1 = 12\,(x_1(1) - s_f) - 6\,(x_2(1) - v_f)\,,$$

$$c_2 = -6\,(x_1(1) - s_f) + 2\,(x_2(1) - v_f)\,,$$

and $x_1(1)$ and $x_2(1)$ are obtained by solving the IVP

$$\dot{x}_1(t) = x_2(t)\,, \quad x_1(0) = s_0\,,$$

$$\dot{x}_2(t) = u^-(t)\,, \quad x_2(0) = v_0\,,$$

for all $t \in [0,1]$.

Projections

Proposition 2 (Projection onto $\mathcal{B}$)**.** *The projection* $P_{\mathcal{B}}$ *of* $u^- \in L^2(0,1;\mathbb{R}^n)$ *onto the constraint set* $\mathcal{B}$*, as the solution of Problem* (P2)*, is given by*

$$P_{\mathcal{B}}(u^-)(t) = \begin{cases} u^-(t)\,, & \text{if } \|u^-(t)\|_2 \leq a\,, \\ a\,\dfrac{u^-(t)}{\|u^-(t)\|_2}\,, & \text{if } \|u^-(t)\|_2 > a\,, \end{cases}$$

for all $t \in [0,1]$.

Best Approximation Algorithms

X is a real Hilbert space

with inner product $\langle \cdot, \cdot \rangle$, induced norm $\| \cdot \|$.

A is a closed affine subspace of X, and B is a nonempty closed convex subset of X.

Given $z \in X$, our aim is to find

$$P_{A \cap B}(z),$$

the projection of z onto the intersection $A \cap B \neq \emptyset$.

Best Approximation Algorithms

We test five methods when $X = L^2(0, 1; \mathbb{R}^n)$, $A = \mathcal{A}$, $B = \mathcal{B}$, and $z = 0$:

- Dykstra's Algorithm [strongly convergent]
  (Boyle–Dykstra 1985)
- Method of Alternating Projections (MAP) [weakly convergent]
  (von Neumann 1948, Bregman 1965)
- Douglas–Rachford (DR) Algorithm [weakly convergent]
  (Douglas–Rachford 1956, Lions–Mercier 1979, Eckstein–Bertsekas 1992)
- Aragón Artacho–Campoy (AAC) Algorithm [strongly convergent]
  (Aragón Artacho–Campoy 2018, Alwadani–Bauschke–Moursi–Wang 2018)
- Fast Iterative Shrinkage-thresholding Algorithm (FISTA) [strong. conv.]
  (Beck–Teboulle 2009, Attouch–Cabot 2018, Bauschke–Bui–Wang 2019)

Best Approximation Algorithms

Algorithm 1 (Dykstra)

Step 1 (*Initialization*) Choose the initial iterates $u^0 = 0$ and $q^0 = 0$. Choose a small parameter $\varepsilon > 0$, and set $k = 0$.

Step 2 (*Projection onto* $\mathcal{B}$) Set $u^- = u^k + q^k$. Compute $\widetilde{u} = P_{\mathcal{B}}(u^-)$.

Step 3 (*Projection onto* $\mathcal{A}$) Set $u^- := \widetilde{u}$. Compute $\widehat{u} = P_{\mathcal{A}}(u^-)$.

Step 4 (*Update*) Set $u^{k+1} := \widehat{u}$ and $q^{k+1} := u^k + q^k - \widetilde{u}$.

Step 5 (*Stopping criterion*) If $\|u^{k+1} - u^k\|_{L^\infty} \leq \varepsilon$, then return $\widetilde{u}$ and stop. Otherwise, set $k := k + 1$ and go to Step 2.

Best Approximation Algorithms

Algorithm 2 (MAP)

Step 1 (*Initialization*) Choose the initial iterate $u^0 = 0$.
Choose a small parameter $\varepsilon > 0$, and set $k = 0$.

Step 2 (*Projection onto $\mathcal{B}$*) Set $u^- = u^k$. Compute $\widetilde{u} = P_{\mathcal{B}}(u^-)$.

Step 3 (*Projection onto $\mathcal{A}$*) Set $u^- := \widetilde{u}$. Compute $\widehat{u} = P_{\mathcal{A}}(u^-)$.

Step 4 (*Update*) Set $u^{k+1} := \widehat{u}$.

Step 5 (*Stopping criterion*) If $\|u^{k+1} - u^k\|_{L^\infty} \leq \varepsilon$, then return $\widetilde{u}$ and stop.
Otherwise, set $k := k + 1$ and go to Step 2.

Best Approximation Algorithms

Algorithm 3 (DR)

Step 1 (*Initialization*) Choose a parameter $\lambda \in \,]0,1[$ and the initial iterate u^0 arbitrarily. Choose a small parameter $\varepsilon > 0$, and set $k = 0$.

Step 2 (*Projection onto $\mathcal{B}$*) Set $u^- = \lambda u^k$. Compute $\widetilde{u} = P_{\mathcal{B}}(u^-)$.

Step 3 (*Projection onto $\mathcal{A}$*) Set $u^- := 2\widetilde{u} - u^k$. Compute $\widehat{u} = P_{\mathcal{A}}(u^-)$.

Step 4 (*Update*) Set $u^{k+1} := u^k + \widehat{u} - \widetilde{u}$.

Step 5 (*Stopping criterion*) If $\|u^{k+1} - u^k\|_{L^\infty} \leq \varepsilon$, then return $\widetilde{u}$ and stop. Otherwise, set $k := k + 1$ and go to Step 2.

Best Approximation Algorithms

Algorithm 4 (AAC)

Step 1 (*Initialization*) Choose the initial iterate u^0 arbitrarily. Choose a small parameter $\varepsilon > 0$, two parameters[1] α and β in $]0, 1[$, and set $k = 0$.

Step 2 (*Projection onto $\mathcal{B}$*) Set $u^- = u^k$. Compute $\widetilde{u} = P_{\mathcal{B}}(u^-)$.

Step 3 (*Projection onto $\mathcal{A}$*) Set $u^- = 2\beta\widetilde{u} - u^k$. Compute $\widehat{u} = P_{\mathcal{A}}(u^-)$.

Step 4 (*Update*) Set $u^{k+1} := u^k + 2\alpha\beta(\widehat{u} - \widetilde{u})$.

Step 5 (*Stopping criterion*) If $\|u^{k+1} - u^k\|_{L^\infty} \leq \varepsilon$, then return $\widetilde{u}$ and stop. Otherwise, set $k := k + 1$ and go to Step 2.

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[1] Aragón Artacho and Campoy recommend $\alpha = 0.9$ and $\beta \in [0.7, 0.8]$ in their paper .

Best Approximation Algorithms

Algorithm 5 (FISTA)

Step 1 (*Initialization*) Choose $\widehat{u}_1 = \widehat{u}_2 = v = 0$, $t_0 = 1$. Choose a small parameter $\varepsilon > 0$, Lipschitz const. $L = 2$ for ℓ_2-norm, and $k = 0$.

Step 2 (*Projection onto* $\mathcal{B}$) Set $u^- := v - L\,\widehat{u}_1$. Compute $\widetilde{u}_1^{k+1} = \widehat{u}_1 - (v - P_{\mathcal{B}}(u^-))/L$.

Step 3 (*Projection onto* $\mathcal{A}$) Set $u^- := v - L\,\widehat{u}_2$. Compute $\widetilde{u}_2^{k+1} = \widehat{u}_2 - (v - P_{\mathcal{A}}(u^-))/L$.

Step 4 (*Update*) Set $u^{k+1} = \widetilde{u}_1^{k+1} + \widetilde{u}_2^{k+1}$.

Step 5 (*Stopping criterion*) If $\|u^{k+1} - u^k\|_{L^\infty} \leq \varepsilon$, then return $\widetilde{u}$ and stop. Otherwise,

set: $t_{k+1} = \frac{1}{2}\left(1 + \sqrt{1 + 4t_k^2}\right)$, $\widehat{u}_i := \widetilde{u}_i^{k+1} + \frac{t_k - 1}{t_{k+1}}\left(\widetilde{u}_i^{k+1} - \widetilde{u}_i^k\right)$, $i = 1, 2$ (Nesterov 1983)

OR $\widehat{u}_i := \widetilde{u}_i^{k+1} + \frac{1-\alpha}{k+1}\left(\widetilde{u}_i^{k+1} - \widetilde{u}_i^k\right)$, $i = 1, 2$, $\alpha > 3$ (Attouch–Cabot 2018)

OR $\widehat{u}_i := \widetilde{u}_i^{k+1} + \frac{1 - \ln^\theta(k+1)}{k+1}\left(\widetilde{u}_i^{k+1} - \widetilde{u}_i^k\right)$, $i = 1, 2$, $\theta > 0$ (Attouch–Cabot 2018)

Set $v := \widehat{u}_1 + \widehat{u}_2$, $k := k + 1$ and go to Step 2.

Numerical Experiments

- Algorithms 1–5 carry out iterations with functions.

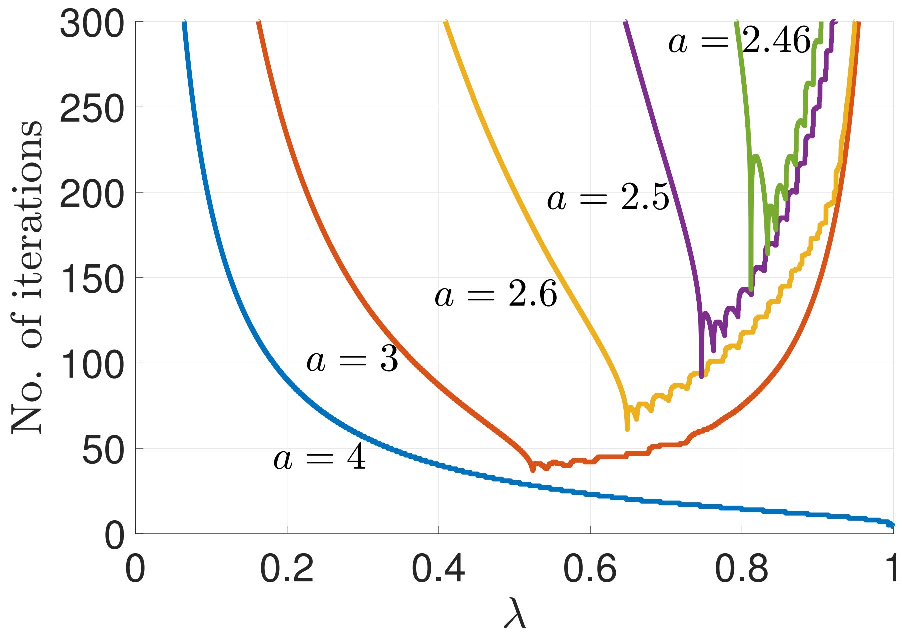
- Use discrete approximations of the functions over the partition $0 = t_0 < t_1 < \ldots < t_N = 1$. For the IVP in computing $P_{\mathcal{A}}$, use Euler's method over the same partition. (Could use any other ODE solver—interested only in $x_i(1)$)

- Define

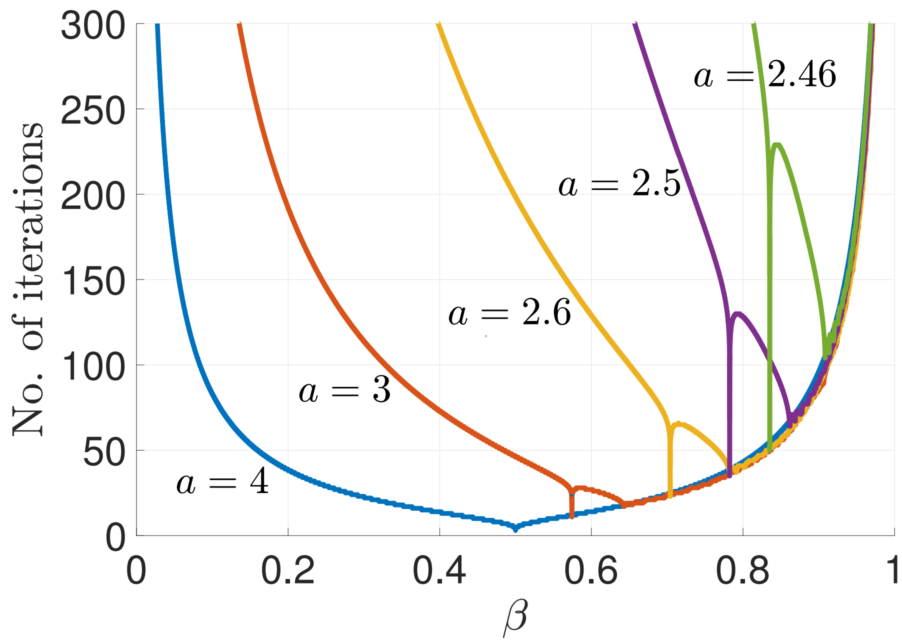
$$\sigma_u^k := \max_{0 \le i \le N-1} |u_i^k - u^*(t_i)| \quad \text{and} \quad \sigma_x^k := \max_{0 \le i \le N} ||x_i^k - x^*(t_i)||_\infty \,.$$

Numerical Experiments ($n = 1$)

Parametric Behaviour



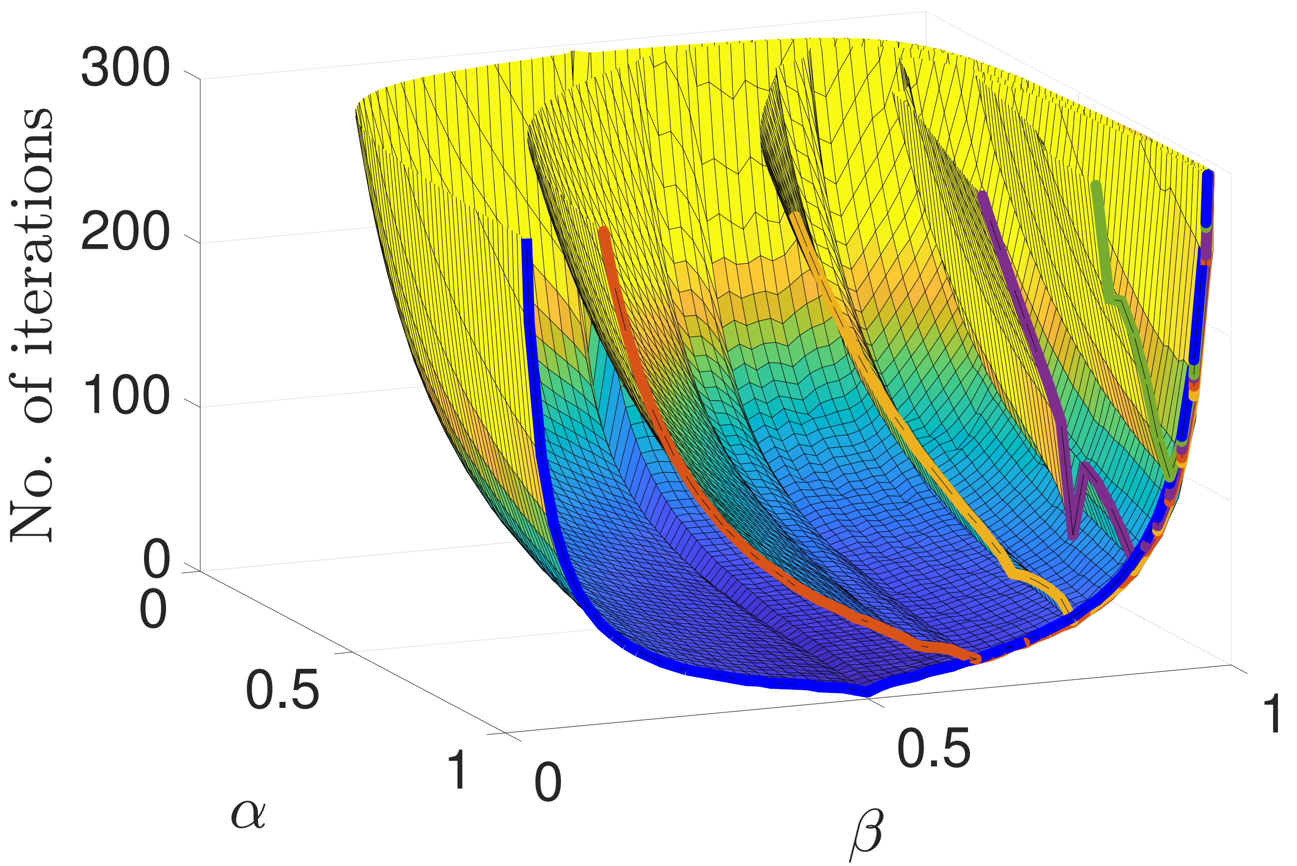
(a) Algorithm 3 (DR)



(b) Algorithm 4 (AAC)
$\alpha = 1$

Numerical Experiments ($n = 1$)

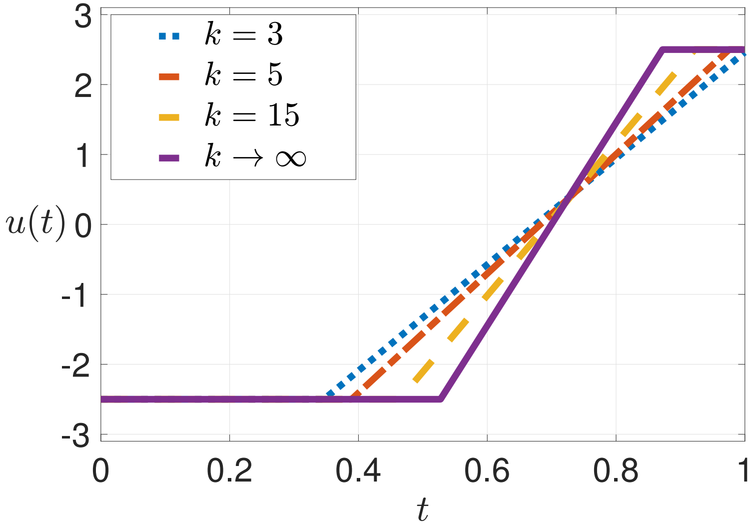
Parametric Behaviour



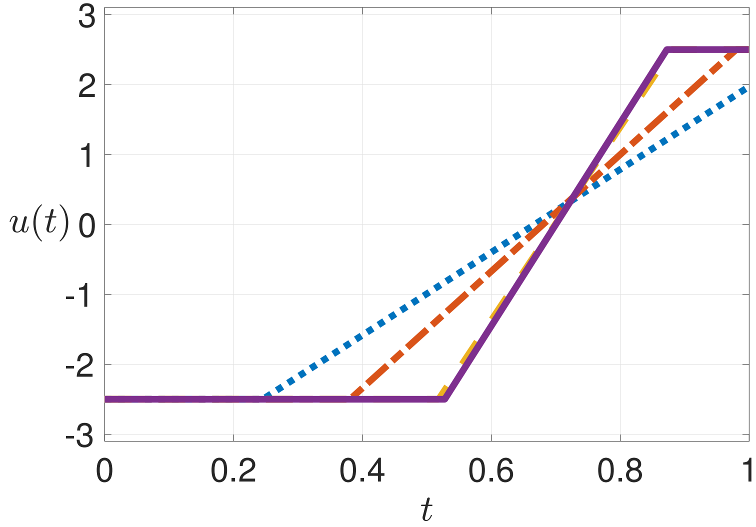
(c) Algorithm 4 (AAC)

Numerical Experiments ($n = 1$)

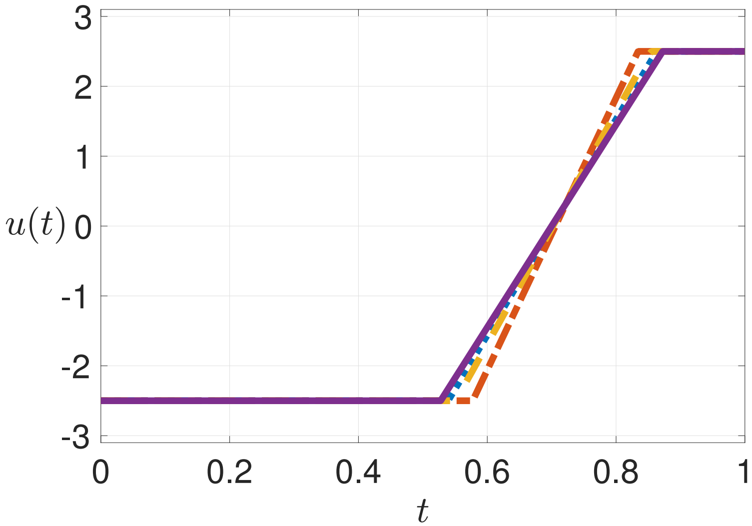
Behaviour in Early Iterations ($N = 2 \times 10^3$)



(a) Algorithm 1 (Dykstra).



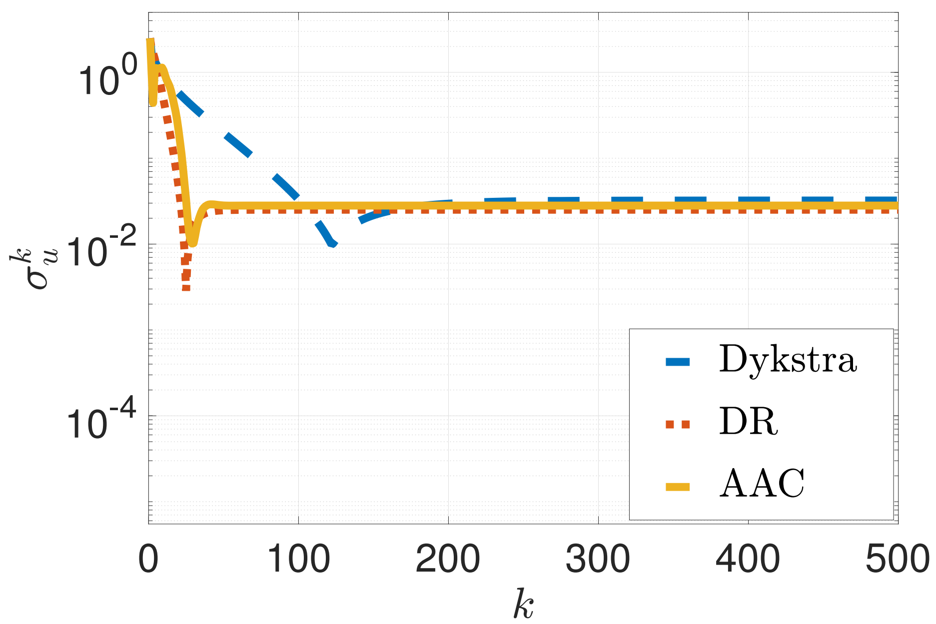
(b) Algorithm 3 (DR, $\lambda = 0.7466$).



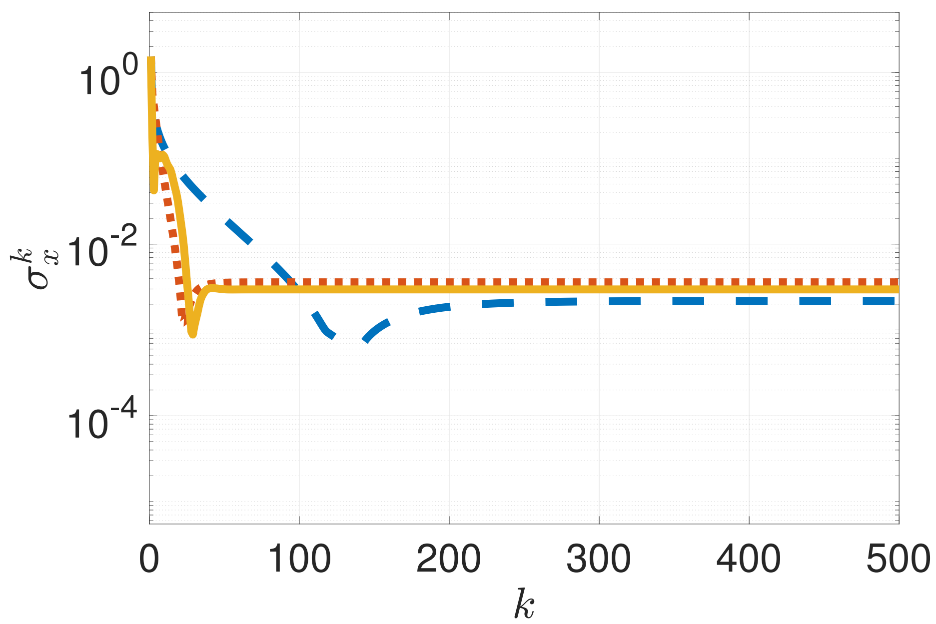
(c) Algorithm 4 (AAC, $\alpha = 1$, $\beta = 0.8617$).

Numerical Experiments ($n = 1$)

Error in Each Iteration



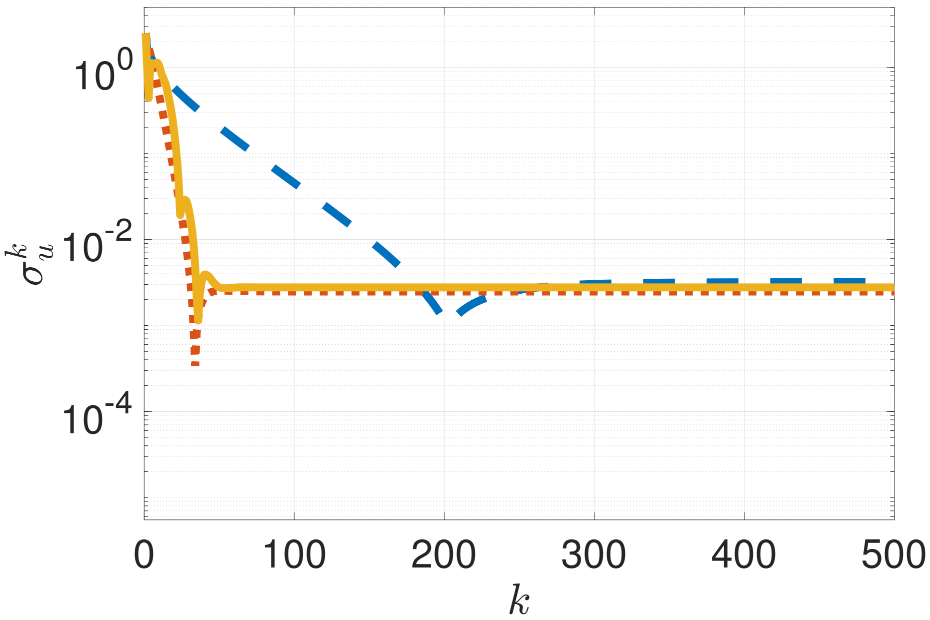
(a) L^∞-error in control with $N = 10^3$.



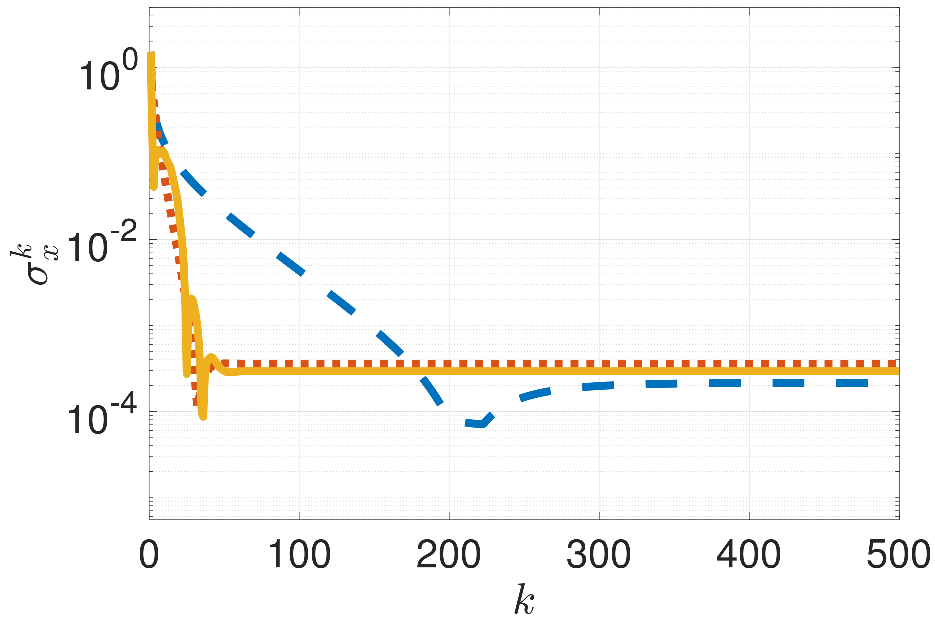
(b) L^∞-error in states with $N = 10^3$.

Numerical Experiments ($n = 1$)

Error in Each Iteration



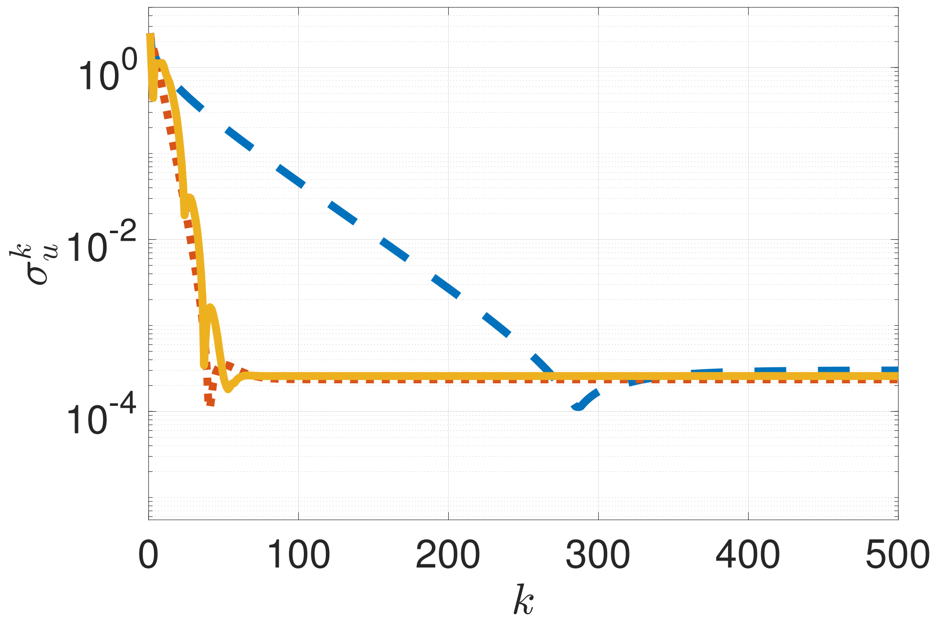
(a) L^∞-error in control with $N = 10^4$.



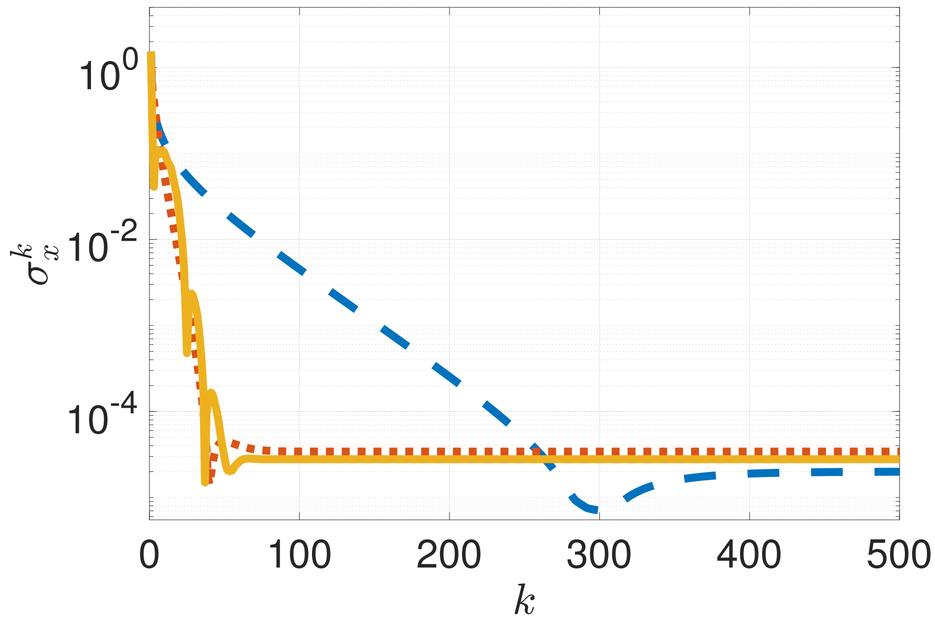
(b) L^∞-error in states with $N = 10^4$.

Numerical Experiments ($n = 1$)

Error in Each Iteration



(a) L^∞-error in control with $N = 10^5$.



(b) L^∞-error in states with $N = 10^5$.

Numerical Experiments ($n = 1$)

N	Dykstra	DR	AAC	Ipopt
10^3	3.2×10^{-2}	2.5×10^{-2}	2.8×10^{-2}	3.2×10^{-2}
10^4	3.2×10^{-3}	2.5×10^{-3}	2.8×10^{-3}	7.7×10^{-3}
10^5	3.0×10^{-4}	2.4×10^{-4}	2.6×10^{-4}	1.6×10^{-2}

(a) L^∞-error in control, σ_u^k .

N	Dykstra	DR	AAC	Ipopt
10^3	2.2×10^{-3}	3.6×10^{-3}	3.0×10^{-3}	2.2×10^{-3}
10^4	2.1×10^{-4}	3.6×10^{-4}	2.9×10^{-4}	2.3×10^{-4}
10^5	2.0×10^{-5}	3.4×10^{-5}	2.8×10^{-5}	8.7×10^{-5}

(b) L^∞-error in states, σ_x^k .

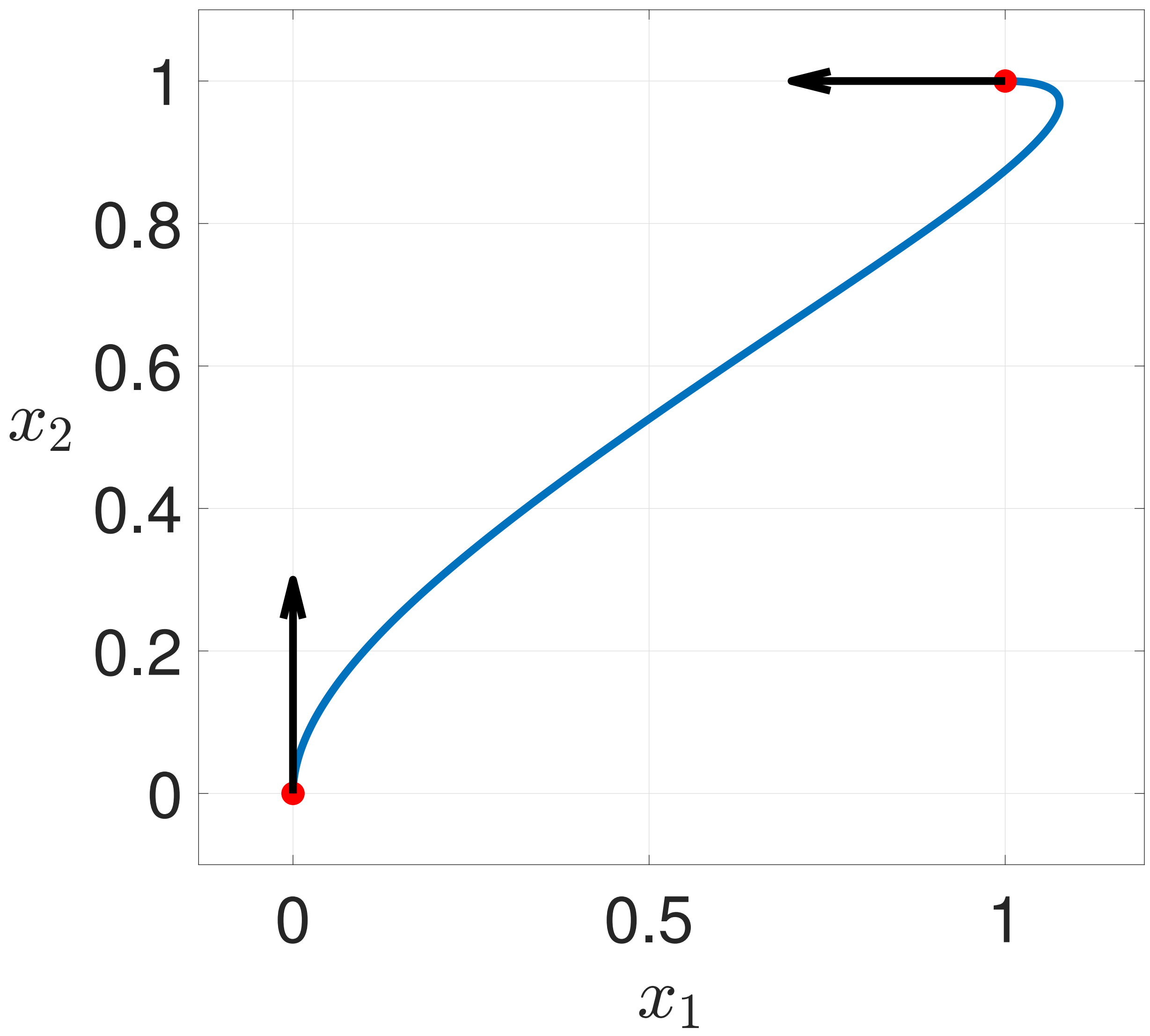
Table 1: Least errors by Algorithms 1, 3–4 and Ipopt ($\varepsilon = 10^{-8}$)

Numerical Experiments ($n = 1$)

N	Dykstra	DR	AAC	Ipopt
10^3	0.03	0.01	0.01	0.08
10^4	0.16	0.05	0.05	0.71
10^5	1.6	0.41	0.28	7.3

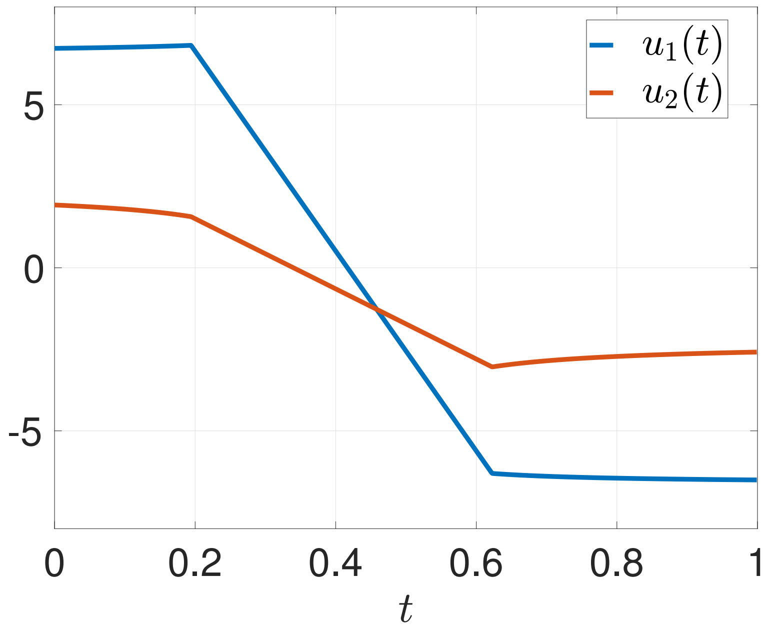
Table 2: CPU times taken by Algorithms 1, 3–4 and Ipopt.

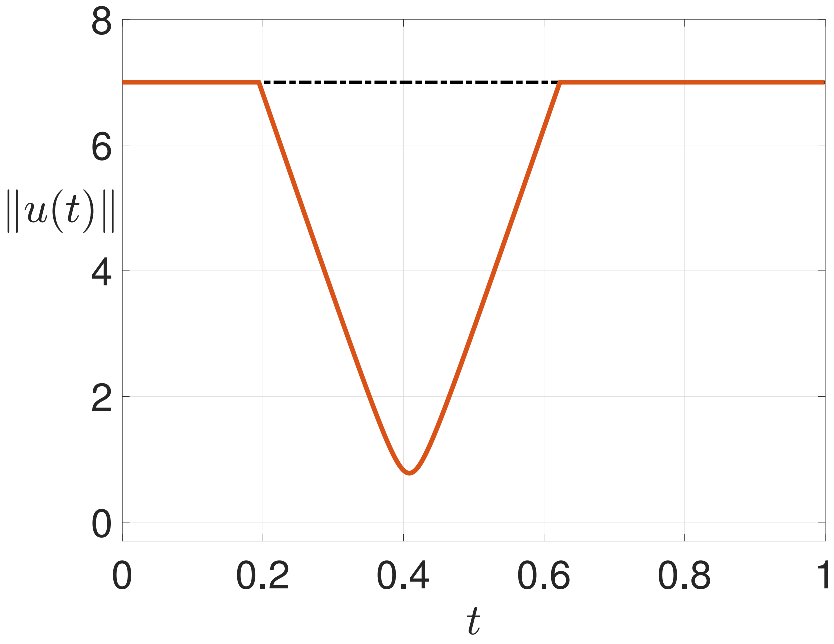
Numerical Experiments ($n = 2$)

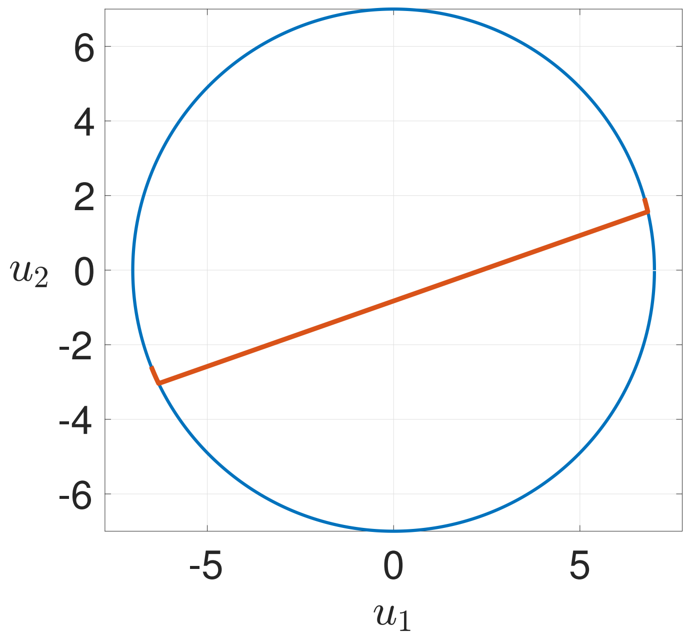


$s_0 = (0, 0), \; v_0 = (0, 1), \quad s_f = (1, 1), \; v_f = (-1, 0)$

Numerical Experiments ($n = 2$)

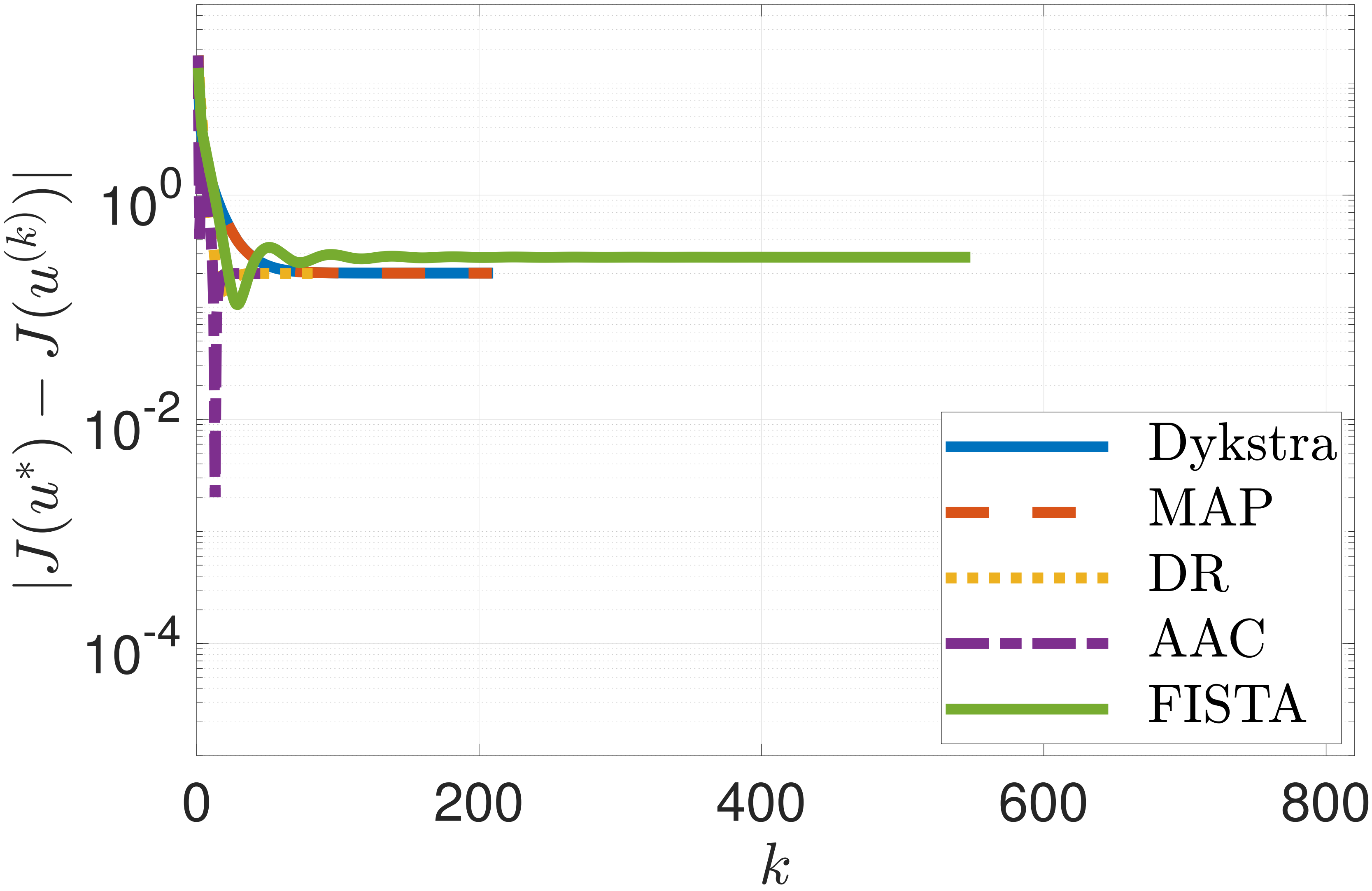






Numerical Experiments ($n = 2$)

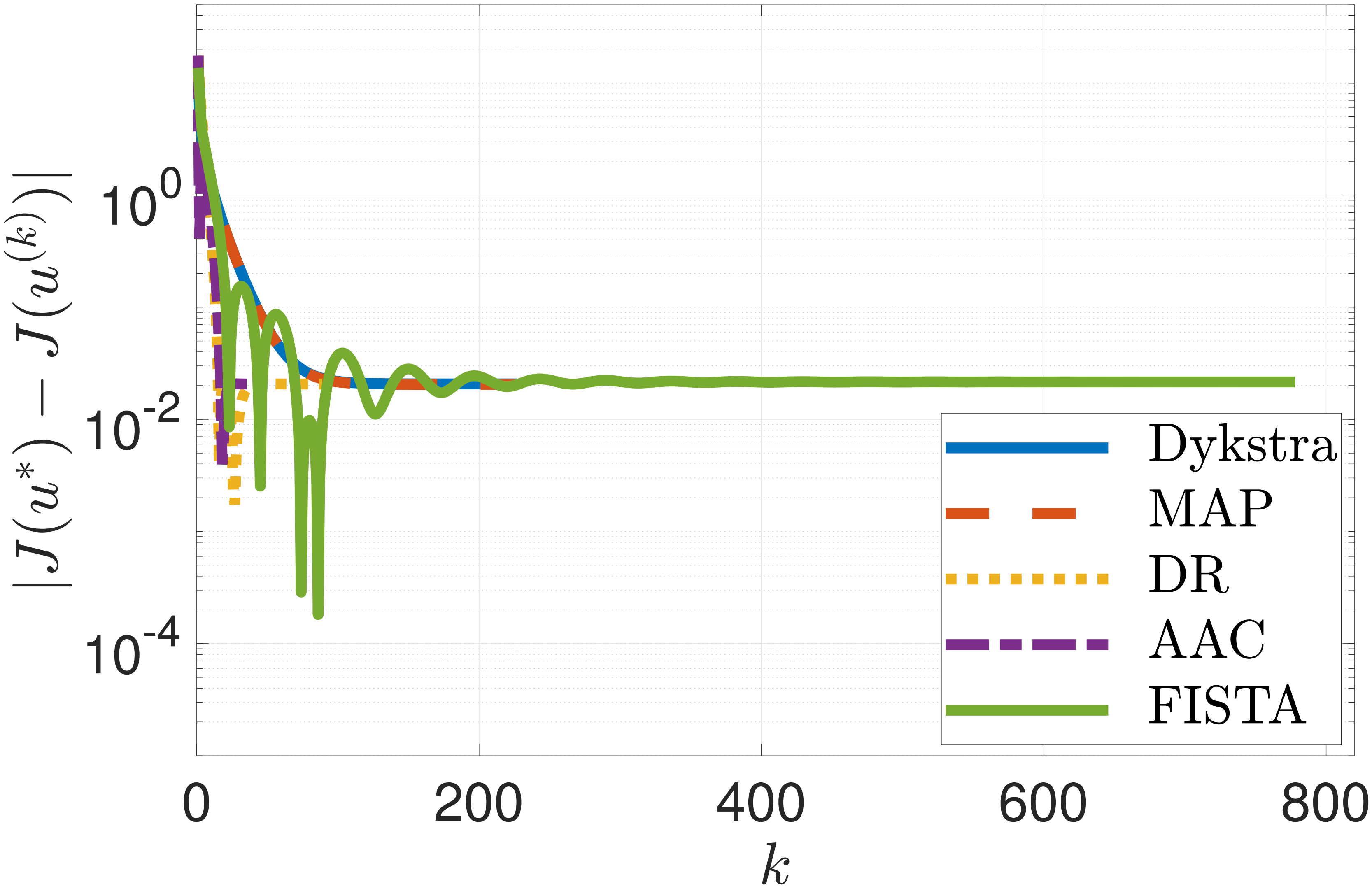
Objective Functional Error in Each Iteration



$N = 10^2$

Numerical Experiments ($n = 2$)

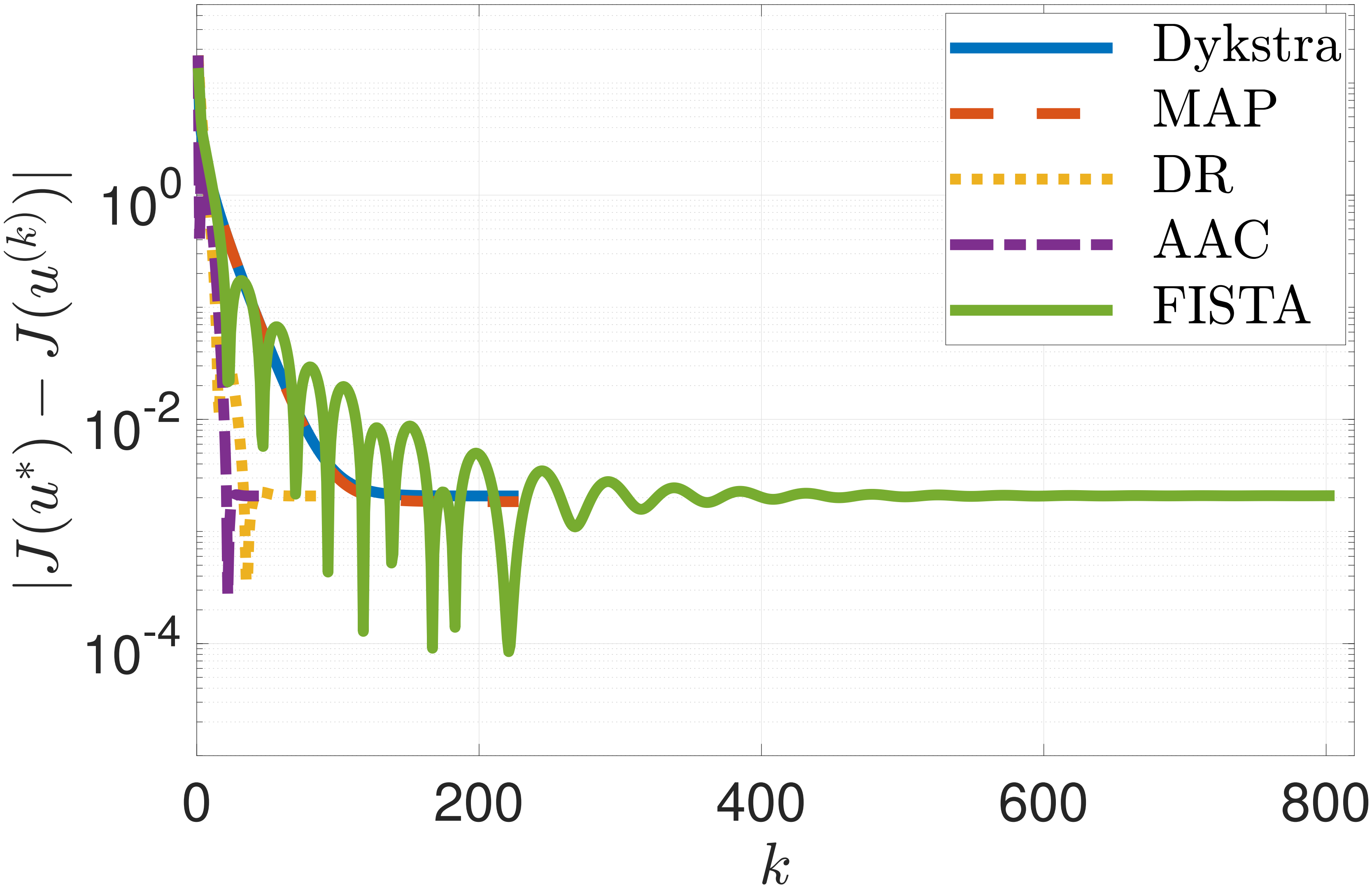
Objective Functional Error in Each Iteration



$N = 10^3$

Numerical Experiments ($n = 2$)

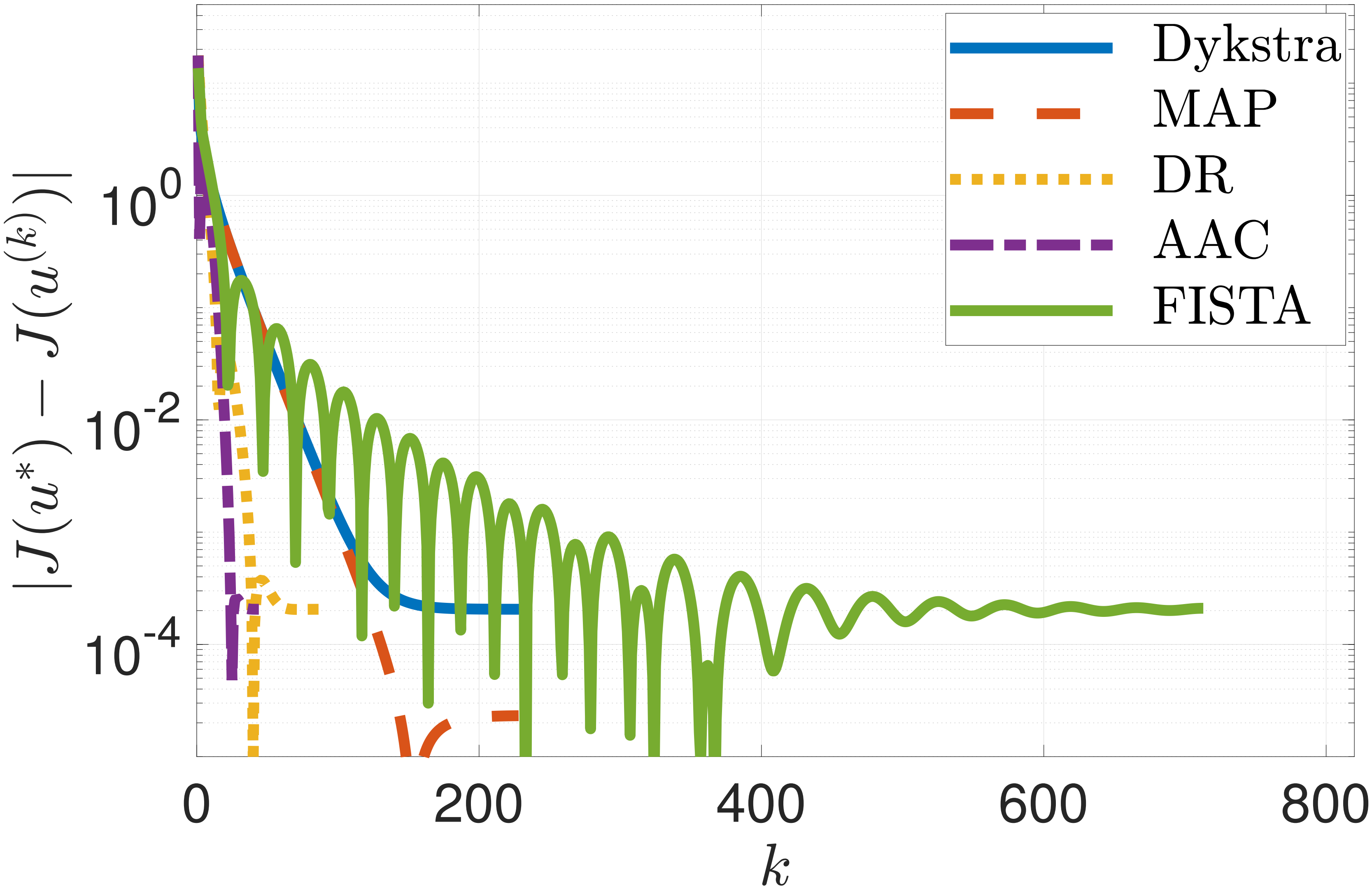
Objective Functional Error in Each Iteration



$N = 10^4$

Numerical Experiments ($n = 2$)

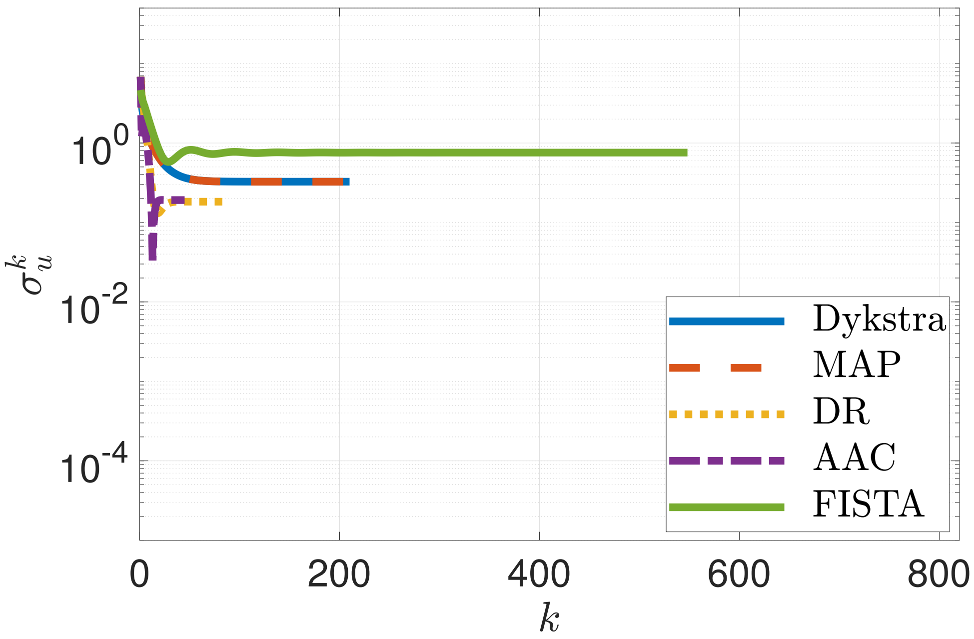
Objective Functional Error in Each Iteration



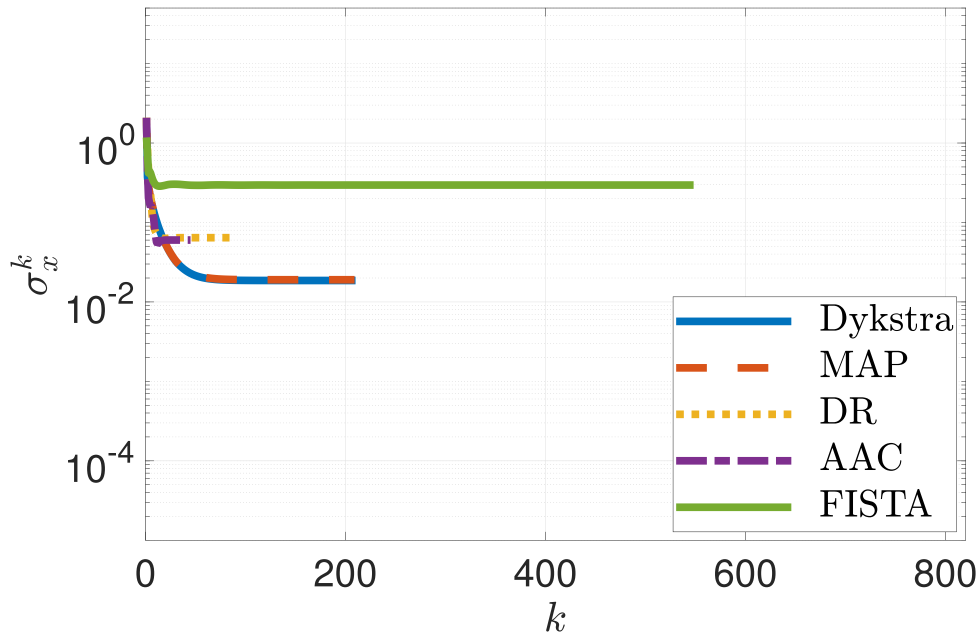
$N = 10^5$

Numerical Experiments ($n = 2$)

State and Control Error in Each Iteration



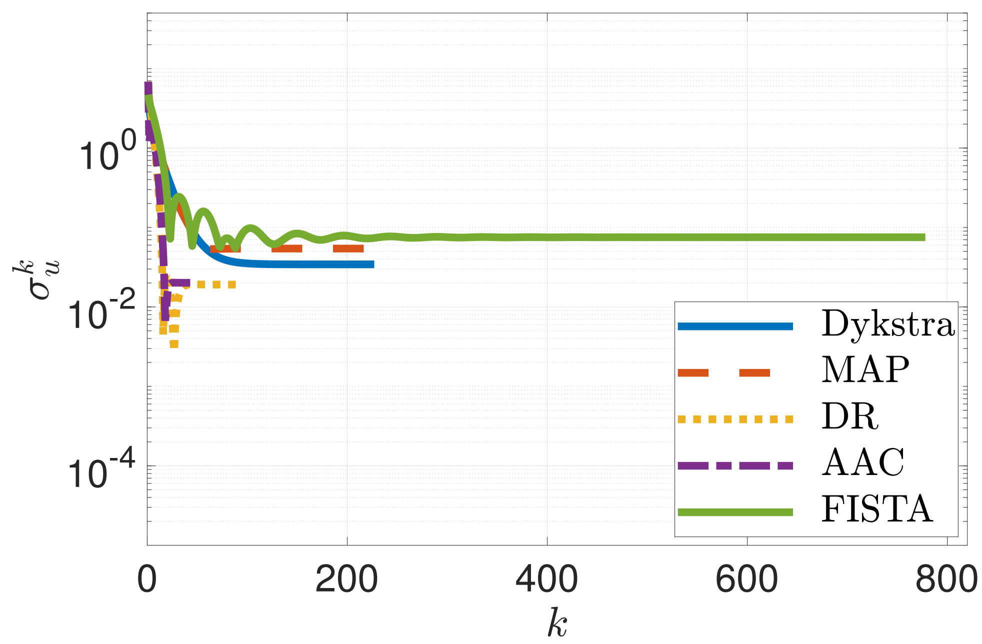
(a) L^∞-error in control with $N = 10^2$.



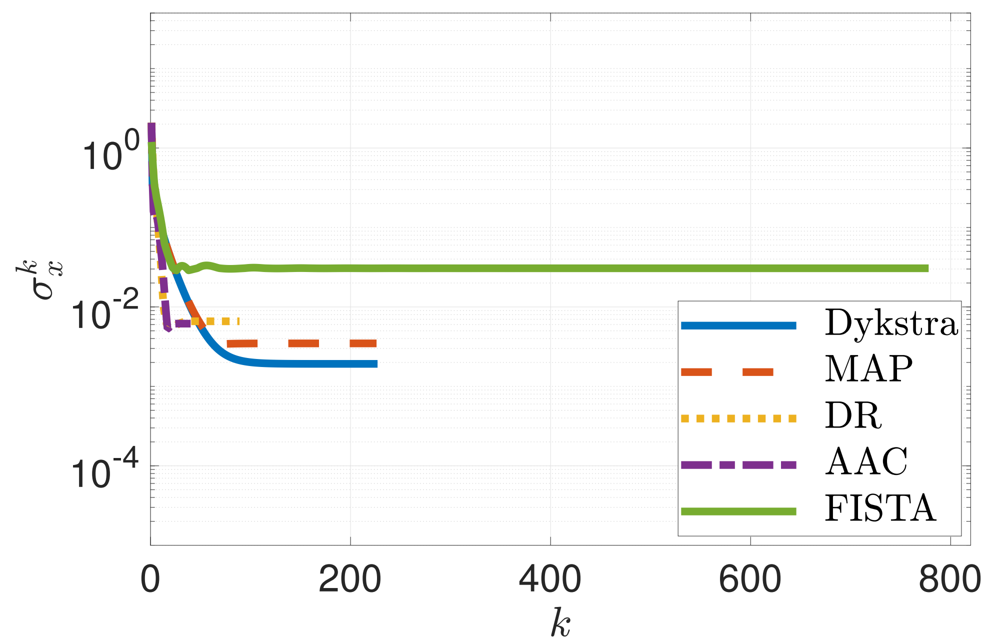
(b) L^∞-error in states with $N = 10^2$.

Numerical Experiments ($n = 2$)

State and Control Error in Each Iteration



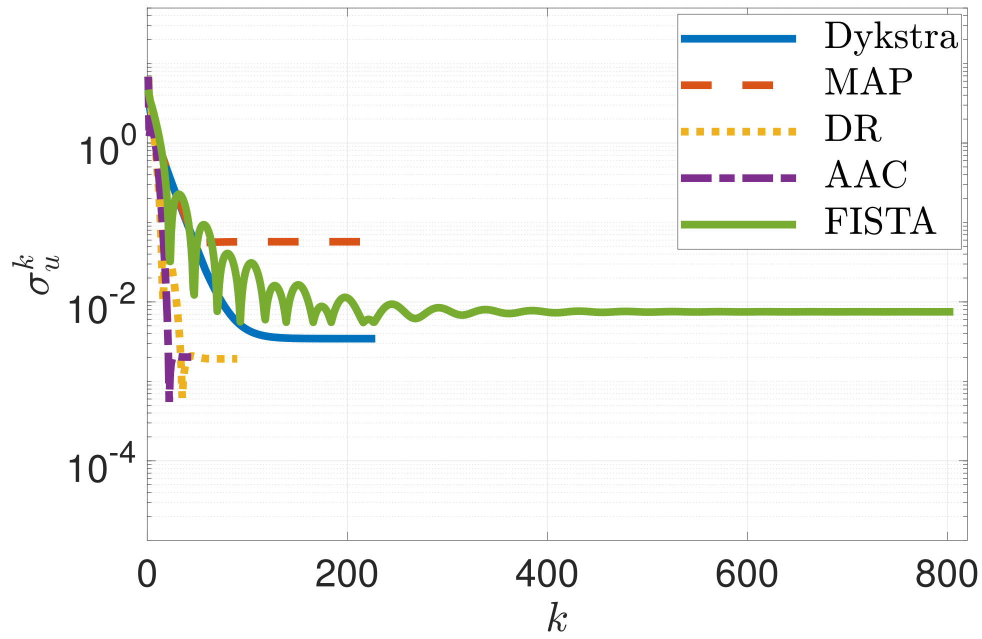
(a) L^∞-error in control with $N = 10^3$.



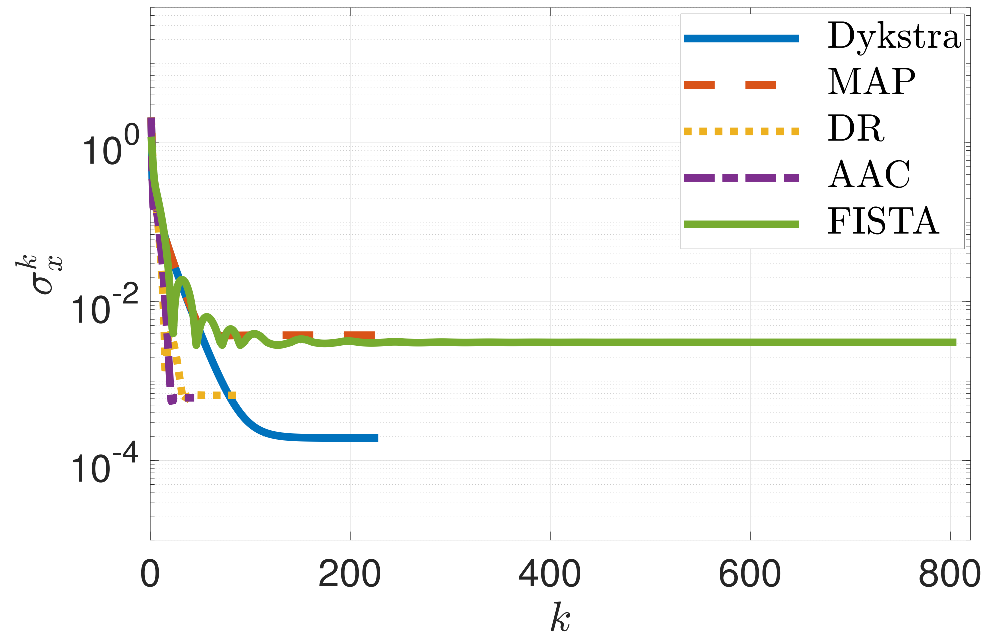
(b) L^∞-error in states with $N = 10^3$.

Numerical Experiments ($n = 2$)

State and Control Error in Each Iteration



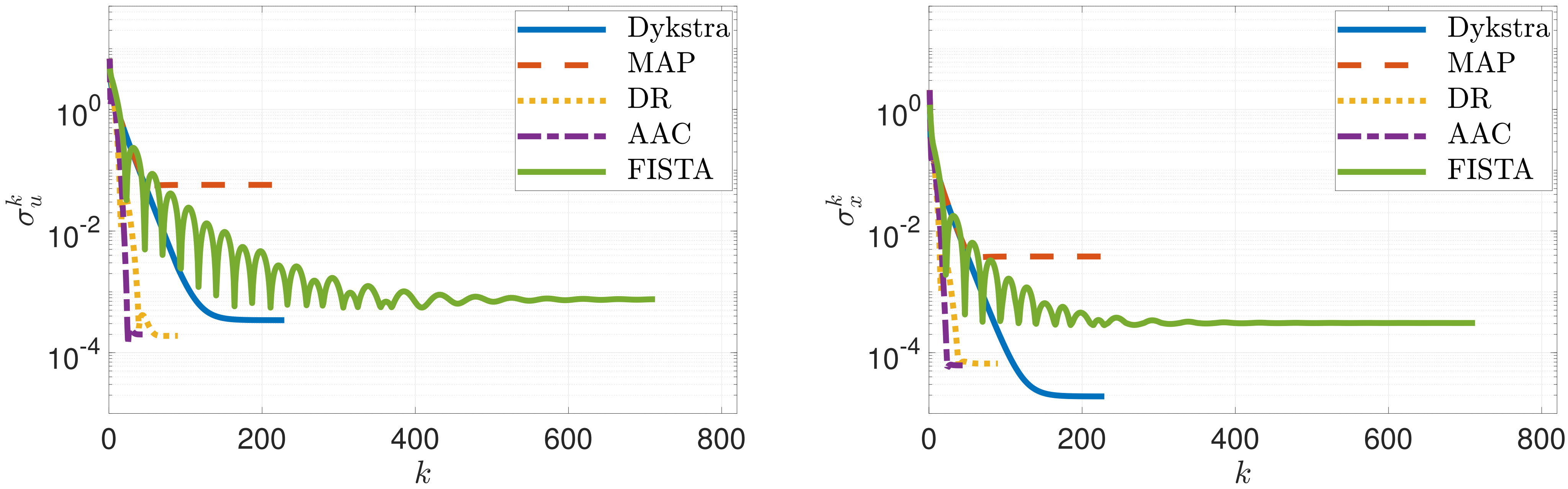
(a) L^∞-error in control with $N = 10^4$.



(b) L^∞-error in states with $N = 10^4$.

Numerical Experiments ($n = 2$)

State and Control Error in Each Iteration



(a) L^∞-error in control with $N = 10^5$.

(b) L^∞-error in states with $N = 10^5$.

Numerical Experiments ($n = 2$)

N	Dykstra	MAP	DR	AAC	FISTA
10^2	3.3×10^{-1}	3.3×10^{-1}	1.8×10^{-1}	1.9×10^{-1}	7.6×10^{-1}
10^3	3.4×10^{-2}	5.4×10^{-2}	1.9×10^{-2}	2.0×10^{-2}	7.5×10^{-2}
10^4	3.4×10^{-3}	5.7×10^{-2}	1.9×10^{-3}	2.0×10^{-3}	7.5×10^{-3}
10^5	3.4×10^{-4}	5.8×10^{-2}	1.9×10^{-4}	2.0×10^{-4}	7.6×10^{-4}

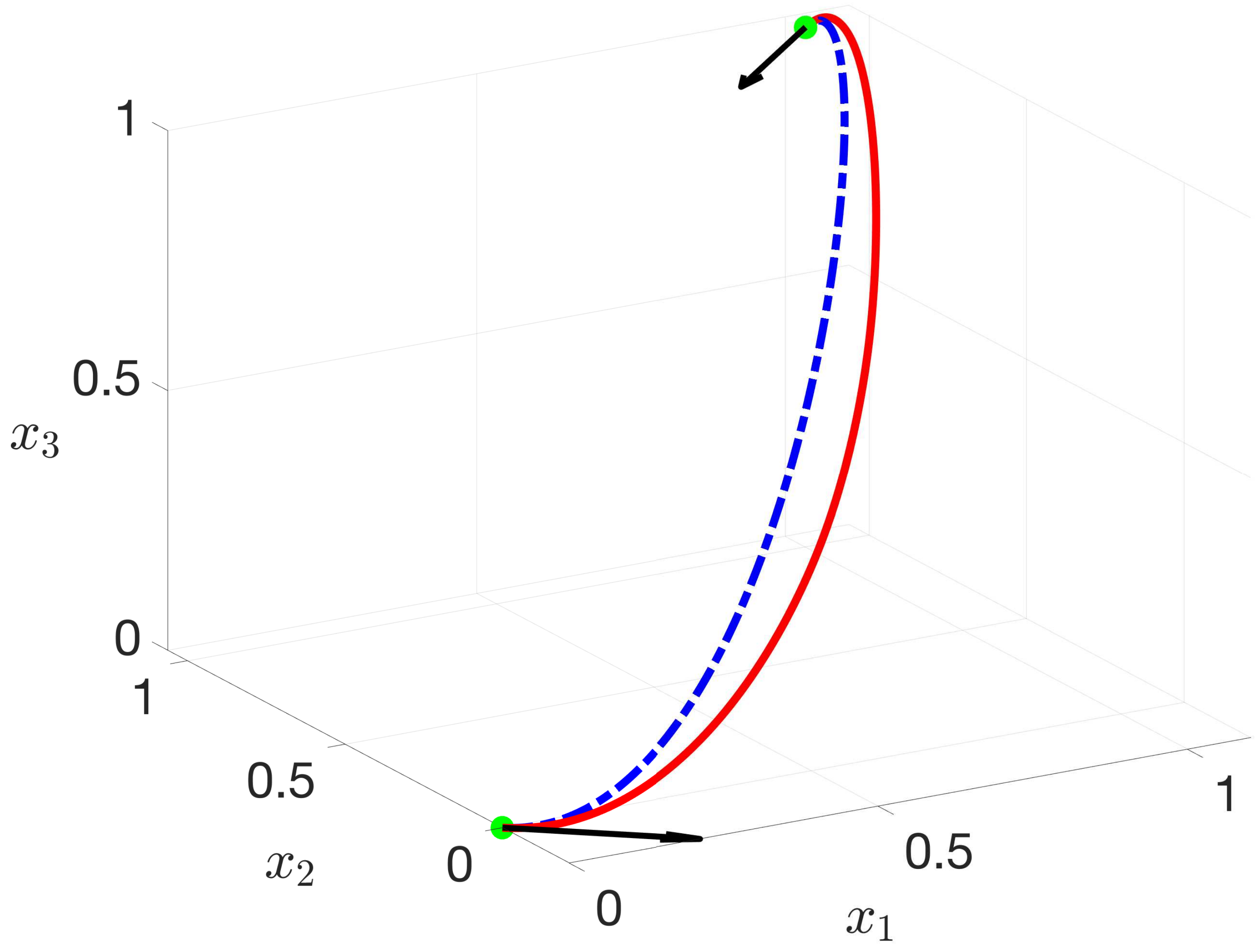
(a) L^∞-error in control, σ_u^k.

N	Dykstra	MAP	DR	AAC	FISTA
10^2	1.9×10^{-2}	1.9×10^{-2}	6.4×10^{-2}	6.0×10^{-2}	3.0×10^{-1}
10^3	1.9×10^{-3}	3.5×10^{-3}	6.6×10^{-3}	6.2×10^{-3}	3.1×10^{-2}
10^4	1.9×10^{-4}	3.8×10^{-3}	6.6×10^{-4}	6.2×10^{-4}	3.1×10^{-3}
10^5	1.9×10^{-5}	3.8×10^{-3}	6.6×10^{-5}	6.2×10^{-5}	3.1×10^{-4}

(b) L^∞-error in states, σ_x^k.

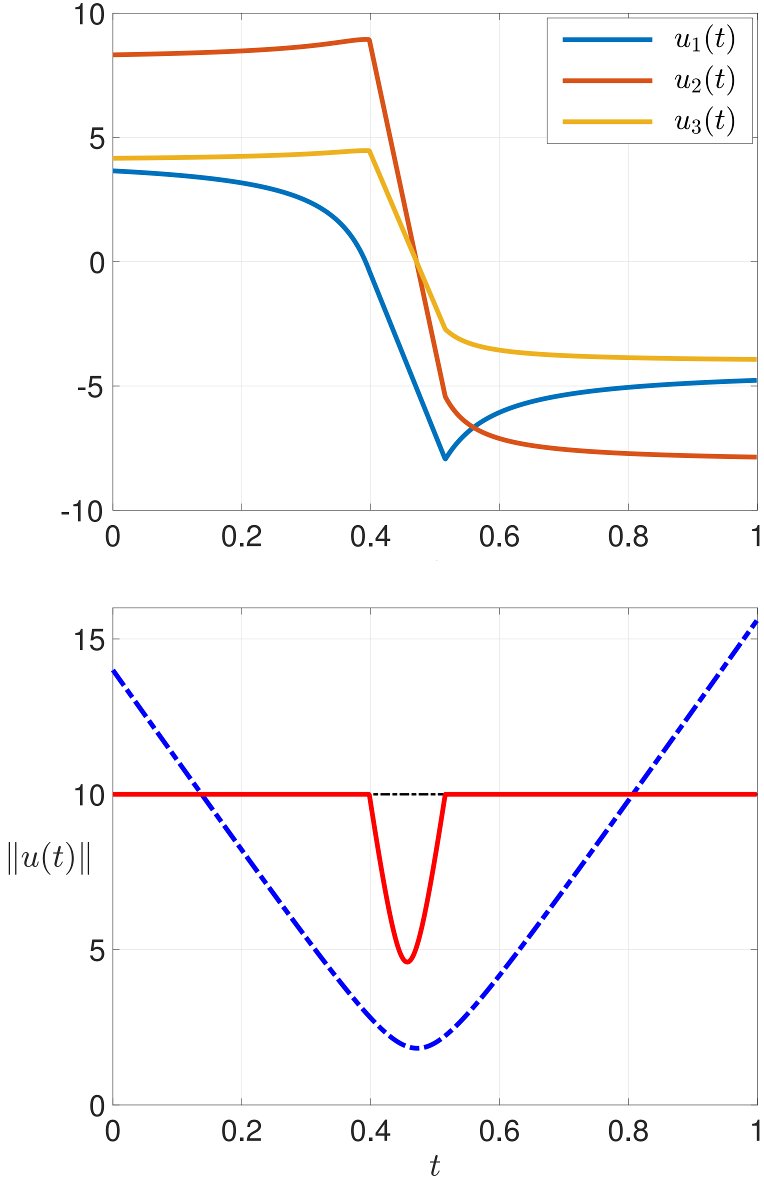
Table 3: Least errors by Algorithms 1–5 ($\varepsilon = 10^{-8}$)

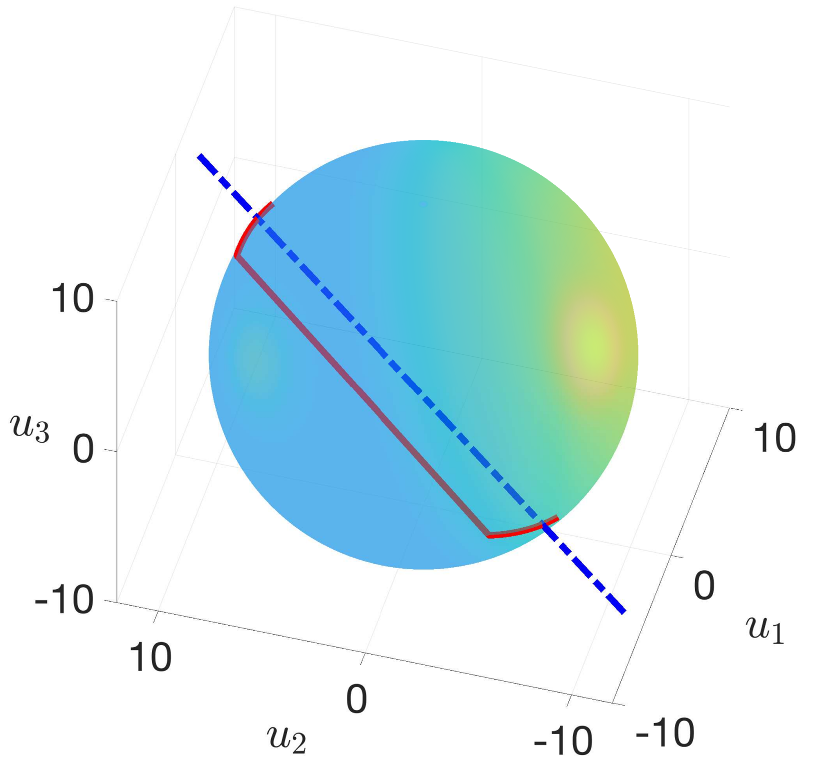
Numerical Experiments ($n = 3$)



$$s_0 = (0, 0, 0), \ \ v_0 = (1, -1, 0), \quad s_f = (1, 1, 1), \ \ v_f = (-1, -1, 0)$$

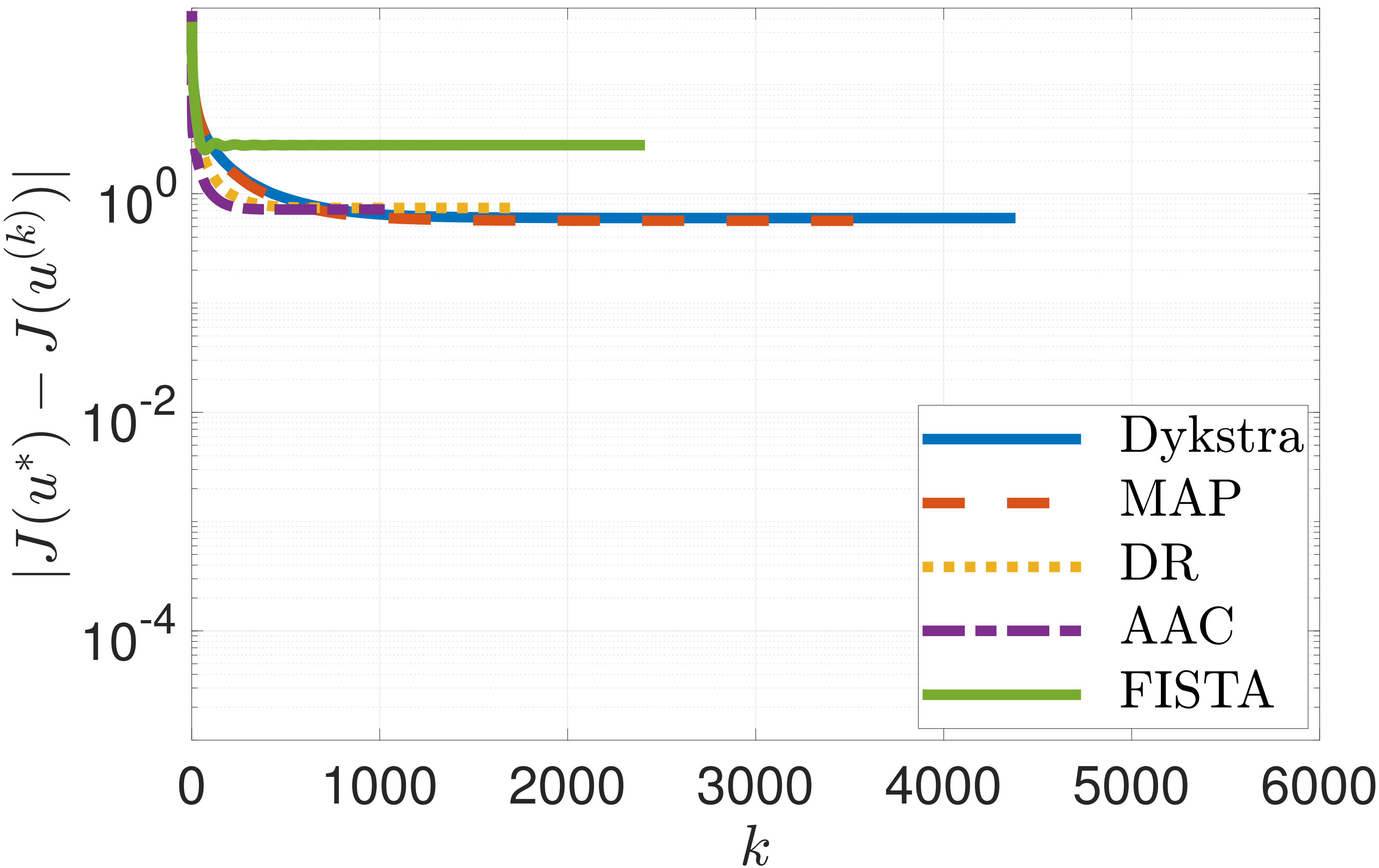
Numerical Experiments ($n = 3$)





Numerical Experiments ($n = 3$)

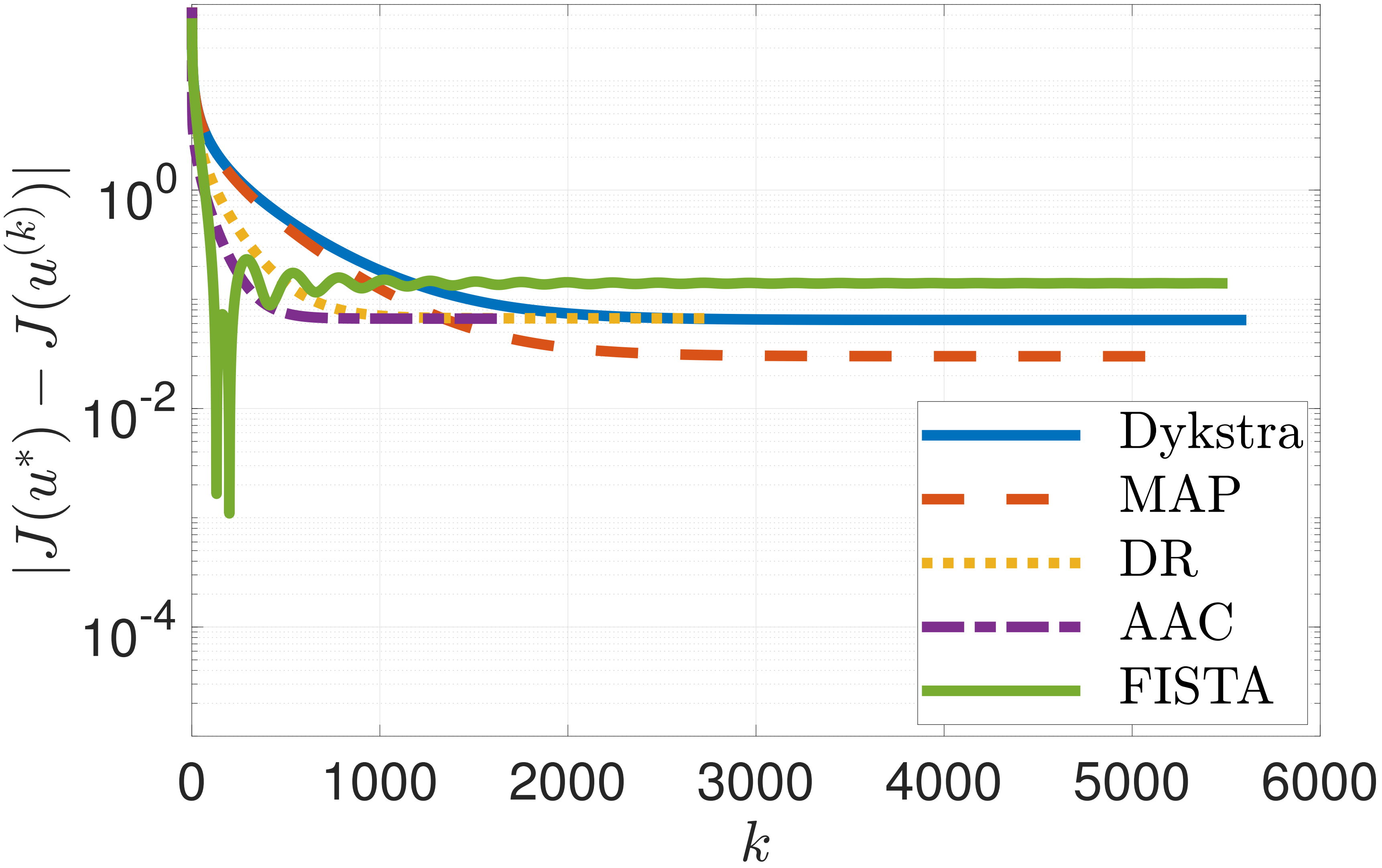
Objective Functional Error in Each Iteration



$N = 10^2$

Numerical Experiments ($n = 3$)

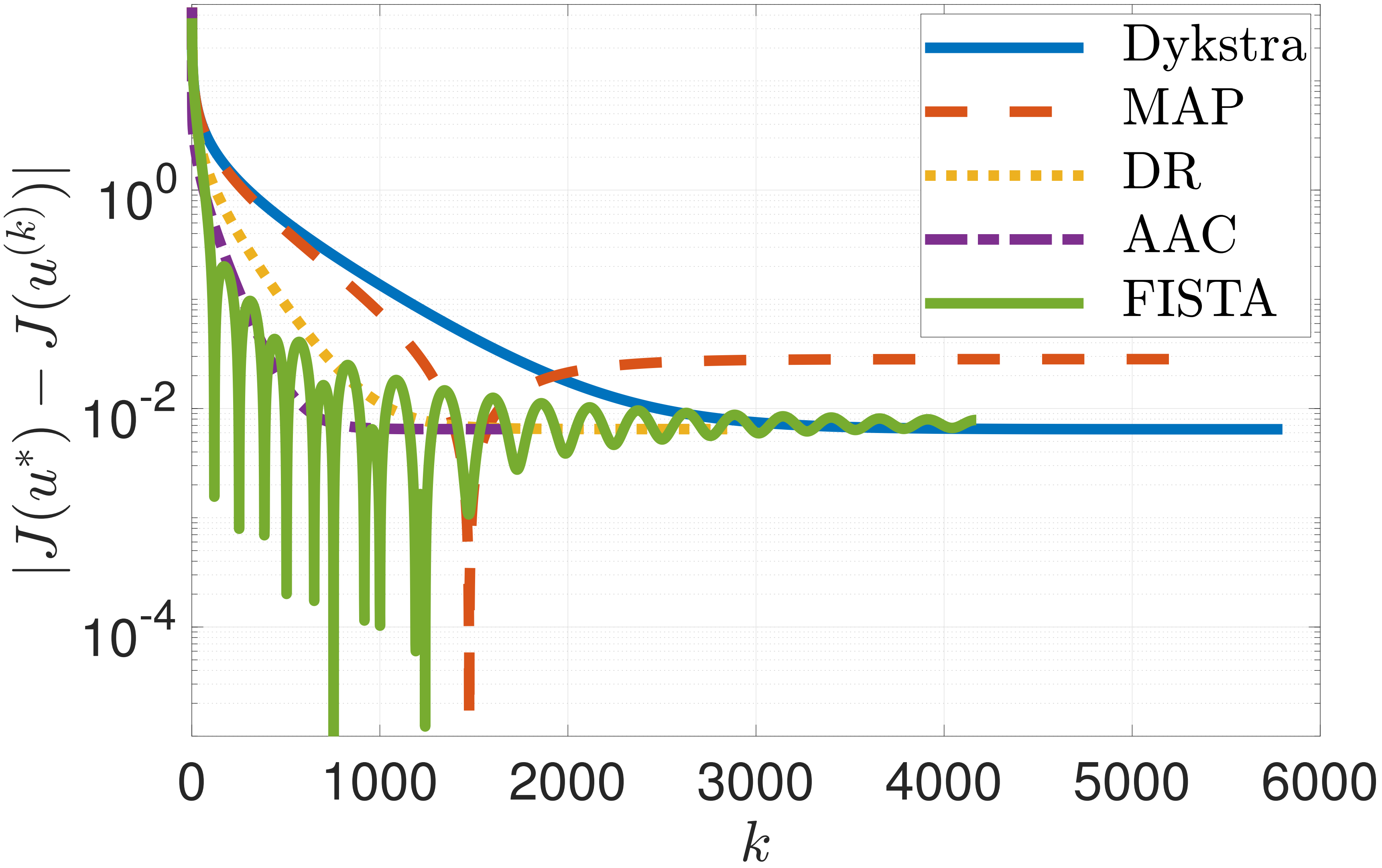
Objective Functional Error in Each Iteration



$N = 10^3$

Numerical Experiments ($n = 3$)

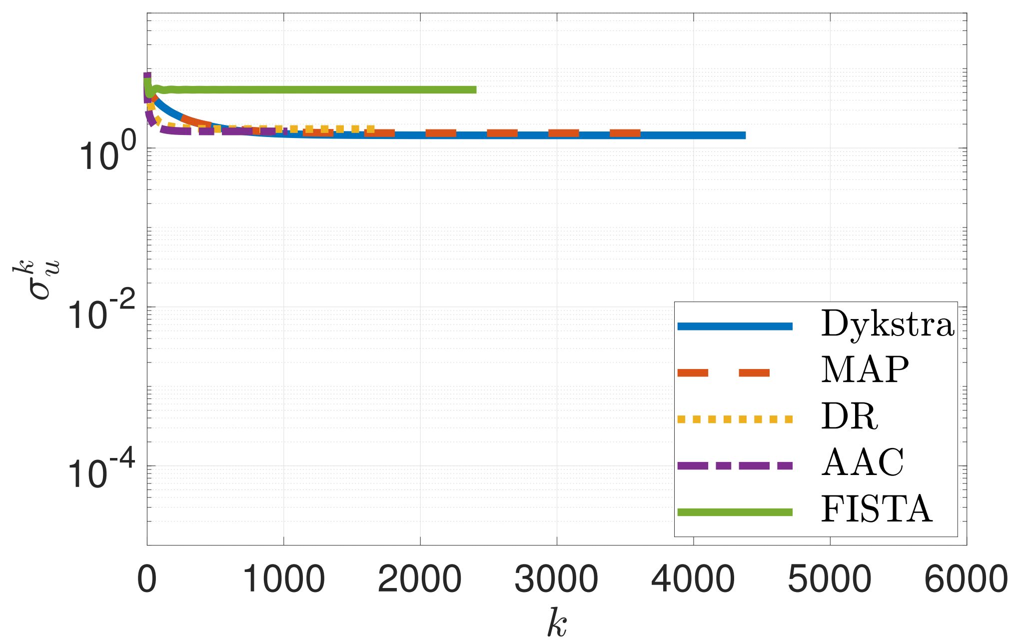
Objective Functional Error in Each Iteration



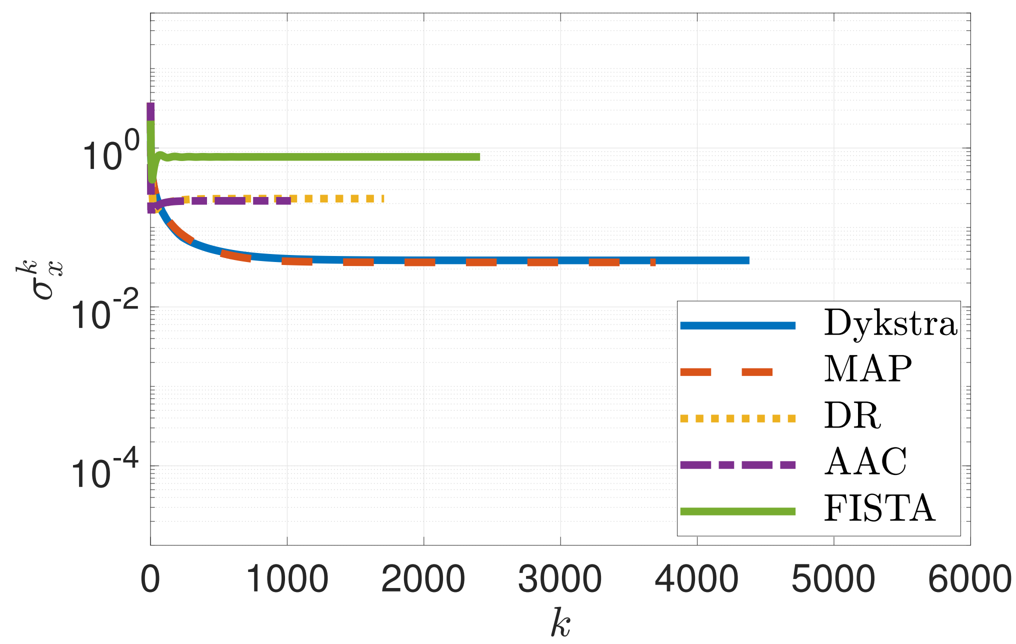
$N = 10^4$

Numerical Experiments ($n = 3$)

State and Control Error in Each Iteration



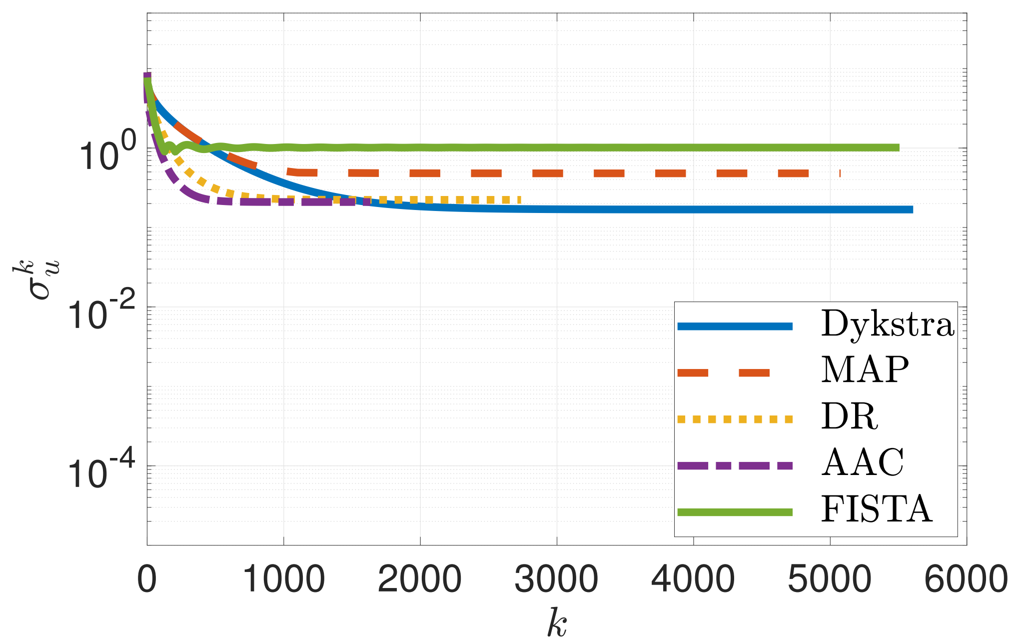
(a) L^∞-error in control with $N = 10^2$.



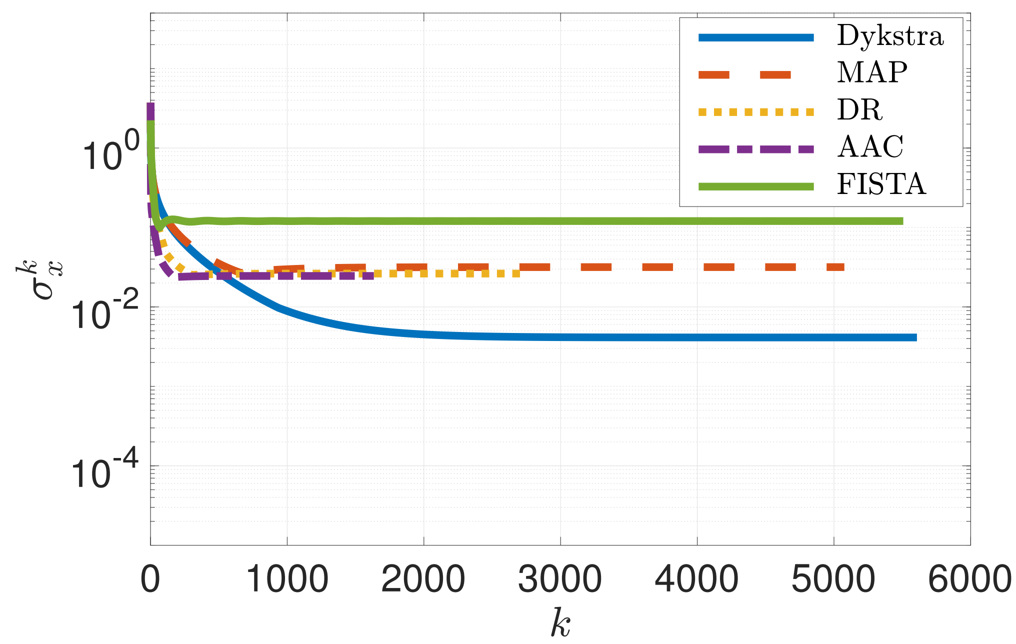
(b) L^∞-error in states with $N = 10^2$.

Numerical Experiments ($n = 3$)

State and Control Error in Each Iteration



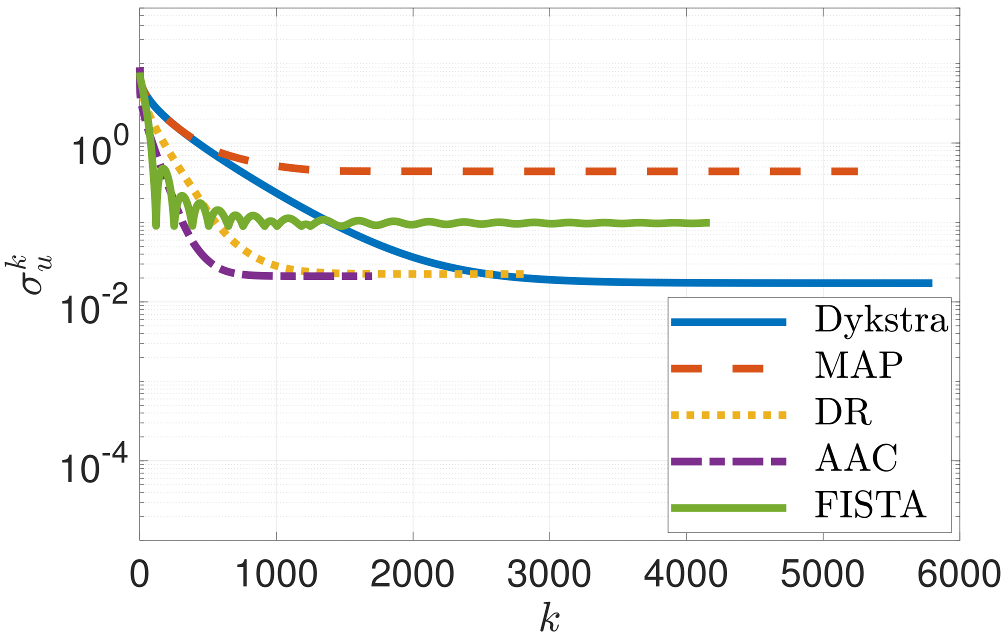
(a) L^∞-error in control with $N = 10^3$.



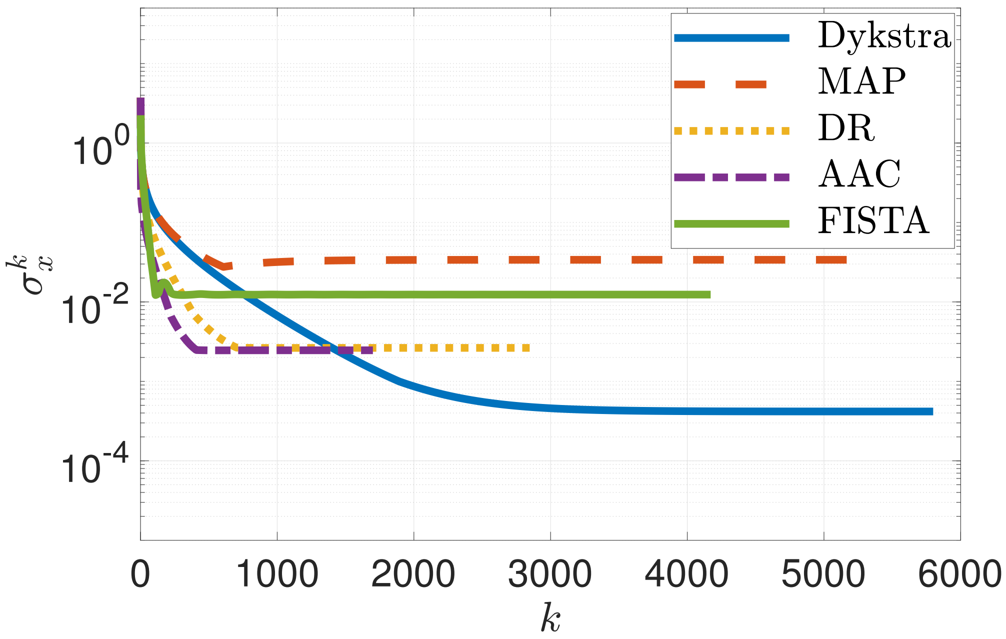
(b) L^∞-error in states with $N = 10^3$.

Numerical Experiments ($n = 3$)

State and Control Error in Each Iteration



(a) L^∞-error in control with $N = 10^4$.



(b) L^∞-error in states with $N = 10^4$.

Numerical Experiments ($n = 3$)

N	Dykstra	MAP	DR	AAC	FISTA
10^2	1.4×10^1	1.5×10^1	1.7×10^1	1.6×10^1	5.4×10^1
10^3	1.7×10^{-1}	4.8×10^{-1}	2.2×10^{-1}	2.1×10^{-1}	1.0×10^1
10^4	1.7×10^{-2}	4.4×10^{-1}	2.2×10^{-3}	2.1×10^{-3}	9.9×10^{-2}

(a) L^∞-error in control, σ_u^k .

N	Dykstra	MAP	DR	AAC	FISTA
10^2	3.9×10^{-2}	3.6×10^{-2}	2.3×10^{-1}	2.2×10^{-1}	7.7×10^{-1}
10^3	4.1×10^{-3}	3.2×10^{-2}	2.6×10^{-2}	2.5×10^{-2}	1.2×10^{-1}
10^4	4.2×10^{-4}	3.4×10^{-2}	2.6×10^{-3}	2.5×10^{-3}	1.2×10^{-2}

(b) L^∞-error in states, σ_x^k .

Table 4: Least errors by Algorithms 1–5 ($\varepsilon = 10^{-8}$)

Conclusion and Open Problems

We observe and note that

- Dykstra, DR, AAC (Algorithms 1, 3 and 4) are the most successful. Dykstra is best in generating optimal states (position and velocity).
- Projection methods are better than the standard discretization approach.
- MAP is observed to converge only weakly for $n = 2$ and 3.
- Models and algorithms here are prototypes for future extensions.

Conclusion and Open Problems

We observe and note that

- Dykstra, DR, AAC (Algorithms 1, 3 and 4) are the most successful. Dykstra is best in generating optimal states (position and velocity).
- Projection methods are better than the standard discretization approach.
- MAP is observed to converge only weakly for $n = 2$ and 3.
- Models and algorithms here are prototypes for future extensions.

Future work

- If $u^-(t)$ is piecewise linear then its projection is piecewise linear. This might simplify further the projection expressions.
- Extension to general control-constrained linear-quadratic problems.
- Extension to nonconvex optimal control problems.