New Constraint Qualifications for Optimization Problems in Banach Spaces based on Asymptotic KKT Conditions

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Basic problem class

$$\begin{aligned} \text{Minimize} \quad & f(x) \\ \text{s.t.} \quad & x \in C \\ & G(x) \in K \end{aligned}$$

Here:

- X, Y are (real) Banach spaces
- $f\colon X \to \mathbb{R}$ and $G\colon X \to Y$ are continuously Fréchet differentiable
- $C \subset X$, $K \subset Y$ are closed and convex (often: cones)

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Generalizes standard NLPs from finite dimensions,
$K = \{0\} \times \hat{K}$ can model equality constraints

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For a convex set $C \subset X$, $\bar{x} \in C$:

$$\mathcal{R}_C(\bar{x}) := \{\lambda\,(x - \bar{x}) \mid x \in C, \lambda \geq 0\} \qquad \text{(radial cone)}$$

$$\mathcal{T}_C(\bar{x}) := \operatorname{cl} \mathcal{R}_C(\bar{x}) \qquad \text{(tangent cone)}$$

$$\mathcal{N}_C(\bar{x}) := (C - \bar{x})^\circ := \{x^\star \in X^\star \mid \langle x^\star, c - \bar{x}\rangle \leq 0 \; \forall x \in C\} \qquad \text{(normal cone)}$$

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For a possible nonconvex set $Q \subset X$, $\bar{x} \in Q$:

$$\mathcal{T}_Q(\bar{x}) := \left\{ d \in X \;\middle|\; \exists (x_k, t_k) \subset Q \times \mathbb{R},\ x_k \to \bar{x},\ t_k \searrow 0,\ \frac{x_k - \bar{x}}{t_k} \to d \right\}$$

$$\mathcal{T}_Q^w(\bar{x}) := \left\{ d \in X \;\middle|\; \exists (x_k, t_k) \subset Q \times \mathbb{R},\ x_k \to \bar{x},\ t_k \searrow 0,\ \frac{x_k - \bar{x}}{t_k} \rightharpoonup d \right\}$$

Towards KKT conditions 1

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If $\bar{x} \in \mathcal{F}$ is a local minimizer, then $-f'(\bar{x}) \in \mathcal{T}_{\mathcal{F}}(\bar{x})^\circ$.

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Proof.

For $d \in \mathcal{T}_{\mathcal{F}}(\bar{x})$, there is (x_k, t_k), $(x_k - \bar{x})/t_k \to d$, and

$$0 \leq \frac{f(x_k) - f(\bar{x})}{t_k}$$

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□

Towards KKT conditions 2

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$$\mathcal{L}_{\mathcal{F}}(\bar{x}) := \left\{ d \in \mathcal{T}_C(\bar{x}) \;\middle|\; G'(\bar{x})d \in \mathcal{T}_K(G(\bar{x})) \right\}$$

and compute its polar cone (i.e. Lemma of Farkas in infinite dimensions):

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Theorem (Kurcyusz (1976), Schirotzek (2007) via Krein-Šmulian)

If $G'(\bar{x})\,\mathcal{T}_C(\bar{x}) - \mathcal{T}_K(G(\bar{x})) = Y$, *then*

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Side note: Can also be proved by sum rule for convex subdifferentials under the Attouch-Brézis condition or via the open mapping theorem by Zowe/Kurcyusz

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Theorem (Robinson (1976))

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Theorem (Zowe, Kurcyusz (1976))

The above assumption is equivalent to

$$Y = G'(\bar{x})\,\mathcal{R}_C(\bar{x}) - \mathcal{R}_K(G(\bar{x})).$$

If $\bar{x}$ *is a local minimizer, then there exist* $\lambda \in \mathcal{N}_K(G(\bar{x}))$ *and* $\mu \in \mathcal{N}_C(\bar{x})$ *such that*

$$0 = L'_x(\bar{x}, \lambda, \mu) = f'(\bar{x}) + G'(\bar{x})^\star \lambda + \mu.$$

Lagrangian: $L(x, \lambda, \mu) = f(x) + \langle \lambda, G(x) \rangle_Y + \mu$

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Definition (KKT point)

A feasible point $\bar{x}$ is called *KKT point*, if there exist $\lambda \in \mathcal{N}_K(G(\bar{x}))$ and $\mu \in \mathcal{N}_C(\bar{x})$ such that

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Definition (Asymptotic KKT point, Steck [2018])

A feasible point $\bar{x}$ is called *s-AKKT point*, if there exists sequences $(x_k) \subset C$, $(\lambda_k) \subset Y^\star$, $(\mu_k) \subset \mathcal{N}_C(x_k)$, $(r_k) \subset [0, \infty)$ such that

$$x_k \to \bar{x} \quad \text{(in } X\text{)} \qquad L'_x(x_k, \lambda_k, \mu_k) \to 0 \quad \text{(in } X^\star\text{)}$$
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- Convex case in infinite dimensions: Thibault, 1997
- Finite dimensions: Andreani, Fazzio, Haeser, Martínez, Ramos, Schuverdt, Secchin, Silva, Svaiter 2010–2019
- "Easy" conditions $x_k \in C$, $\mu_k \in \mathcal{N}_C(x_k)$ handled explicitely
- Convergence of multipliers would imply KKT point
- Def. of *w-AKKT point*: only $x_k \rightharpoonup \bar{x}$ and $L'_x(x_k, \lambda_k, \mu_k) \overset{\star}{\rightharpoonup} 0$ (useful for numerical methods)

Theorem (Börgens, Kanzow, Mehlitz, Wachsmuth (2020))

Assume: $\bar{x}$ *local minimizer,* X *reflexive,* f *weakly sequentially lsc and*

$$\forall (x_k) \subset C, x \in C: \quad x_k \rightharpoonup x \text{ and } d_K(G(x_k)) \to 0 \;\Rightarrow\; G(x) \in K \qquad (*)$$

Then, $\bar{x}$ *is a s-AKKT point.*

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Idea of the proof

Apply Ekeland's variational principle yields minimizers x_k of

$$\min\ f(x) + \|x - \bar{x}\|_X^2 + k\, d_K^2(G(x)) + \frac{1}{k}\,\|x - x_k\|_X \quad \text{w.r.t. } x \in B_r(\bar{x}) \cap C.$$

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Apply optimality conditions via Clarke's subdifferential

$$X := \mathbb{R} \times L^2(0,1), \qquad Y := L^2(0,1), \qquad K := \{0\} \subset Y,$$
$$C := \mathbb{R} \times \{u \in L^2(0,1) \mid -1 \le u \le 1\}, \qquad G(\alpha, u) := \alpha \cdot q - u,$$

where $q \in L^2(0,1) \setminus L^\infty(0,1)$ is fixed, f arbitrary.

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$$\mathcal{F} = \{0\}, \quad \mathcal{T}_{\mathcal{F}}(0)^\circ = X^\star \ne G'(0)^\star \mathcal{T}_C(0)^\circ + \mathcal{T}_K(0)^\circ \quad \text{(codimension 1)}$$

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- Depending on f, 0 might not be a KKT point (even if f is linear!)
- 0 is always a s-AKKT point

Example 2

$$X := \ell^1, \qquad Y := \ell^2, \qquad C := \ell^1,$$

$$K := \{0\} \subset \ell^2, \qquad G(x) := x, \qquad f(x) := \sum_{i=1}^{\infty} a_i \, x_i$$

for some given sequence $a \in \ell^\infty \setminus c_0$. $\bar{x} := 0$ is the only feasible point, therefore optimal.

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No s-AKKT point: otherwise

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$\Rightarrow$ Reflexivity of X is important!

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We are looking for *AKKT regularity conditions*, which imply

$$\bar{x} \text{ is s-AKKT point} \quad \Rightarrow \quad \bar{x} \text{ is KKT point}$$

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We note that:

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A set-valued mapping

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Recall that $\mathcal{M}(\bar{x}, 0) = \mathcal{L}_{\mathcal{F}}(\bar{x})^\circ$ requires a CQ (e.g. $\mathcal{M}(\bar{x}, 0)$ is weak-$\star$ closed)

Painlevé–Kuratowski-type outer/upper limits:

$$\limsup_{\substack{x \to \bar{x} \\ r \searrow 0}} \mathcal{M}(x, r) := \left\{ \bar{v} \in X^\star \;\middle|\; \begin{array}{c} \exists((x_k, r_k, v_k)) \subset X \times [0, \infty) \times X^\star\colon \\ x_k \to \bar{x},\ r_k \searrow 0,\ v_k \to \bar{v}, \\ v_k \in \mathcal{M}(x_k, r_k)\, \forall k \in \mathbb{N} \end{array} \right\}$$

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This readily yields:

$$\bar{x} \text{ is s-AKKT point} \quad \Leftrightarrow \quad -f'(\bar{x}) \in \limsup_{\substack{x \to \bar{x} \\ r \searrow 0}} \mathcal{M}(x,r)$$

AKKT regularity

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Definition

Let $\bar{x} \in \mathcal{F}$ be given. This point is called

- s-AKKT regular, if $$\limsup_{\substack{x \to \bar{x} \\ r \searrow 0}} \mathcal{M}(x, r) \subset \mathcal{M}(\bar{x}, 0)$$
- sw-AKKT regular, if $$\text{w}^*\text{-}\limsup_{\substack{x \to \bar{x} \\ r \searrow 0}} \mathcal{M}(x, r) \subset \mathcal{M}(\bar{x}, 0)$$
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sw-AKKT regular $\Longrightarrow$ $\mathcal{M}(\bar{x}, 0) = \mathcal{L}_{\mathcal{F}}(\bar{x})^\circ$

Directly from the definition:

Theorem

Let $\bar{x}$ be feasible.

1. *If $\bar{x}$ is an s-AKKT point and s-AKKT regular, then $\bar{x}$ is a KKT point.*

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There is an analogue for the weak version

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Definition

We say that the *Robinson/Zowe/Kurcyusz constraint qualification* (RZKCQ) holds at a feasible point $\bar{x} \in \mathcal{F}$ if

$$Y = G'(\bar{x})\,\mathcal{R}_C(\bar{x}) - \mathcal{R}_K(G(\bar{x})).$$

Definition of RZKCQ

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- General purpose CQ in infinite dimensions
- For finite-dimensional problems, it reduces to MFCQ
- Implies boundedness of multipliers

Theorem (Börgens, Kanzow, Mehlitz, Wachsmuth (2020))

Assume that RZKCQ holds at a feasible point $\bar{x} \in \mathcal{F}$. Then $\bar{x}$ is sw-AKKT regular (and, thus, s-AKKT regular).

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Idea of the proof.

$$\bar{v} \in \text{w}^*\text{-}\limsup_{\substack{x \to \bar{x} \\ r \searrow 0}} \mathcal{M}(x, r) := \left\{ \bar{v} \in X^\star \;\middle|\; \begin{array}{c} \exists((x_k, r_k, v_k)) \subset X \times [0, \infty) \times X^\star : \\ x_k \to \bar{x},\, r_k \searrow 0,\, v_k \stackrel{\star}{\rightharpoonup} \bar{v}, \\ v_k \in \mathcal{M}(x_k, r_k)\, \forall k \in \mathbb{N} \end{array} \right\}$$

$$\mathcal{M}(x_k, r_k) := \left\{ G'(x_k)^\star \lambda_k + \mu_k \in X^\star \;\middle|\; \begin{array}{l} \lambda_k \in Y^\star,\, \mu_k \in \mathcal{N}_C(x), \\ \langle \lambda_k, y - G(x_k) \rangle_Y \le r_k \;\forall y \in K \end{array} \right\}$$

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We have to show $\bar{v} \in \mathcal{M}(\bar{x}, 0)$. Due to RZKCQ, the sequence of multipliers is bounded. □

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Definition (Abadie CQ)

Let $\bar{x} \in \mathcal{F}$ be a feasible point. We say that *Abadie's constraint qualification* (ACQ) holds at $\bar{x}$ if

$$\mathcal{T}_{\mathcal{F}}(\bar{x}) = \mathcal{L}_{\mathcal{F}}(\bar{x})$$

holds and $\mathcal{M}(\bar{x}, 0) = \mathcal{L}_{\mathcal{F}}(\bar{x})^{\circ}$.

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ACQ implies that local minimizers are KKT points

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Let X be refl. and sep. We need the Fréchet/limiting normal cone:

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Folklore (Rockafellar/Wets): For every $v \in \widehat{\mathcal{N}}_{\mathcal{F}}(\bar{x})$, there is convex, differentiable h such that $\bar{x}$ is local min and $v = -h'(\bar{x})$

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Theorem (Börgens, Kanzow, Mehlitz, Wachsmuth (2020))

Assume that X *is refl., sep.,* $(*)$ *and* $\bar{x}$ *is sw-AKKT regular. Then, ACQ holds.*

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□

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Definition

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Theorem (Börgens, Kanzow, Mehlitz, Wachsmuth (2020))

Let X be refl. and sep. Let $\bar{x} \in \mathcal{F}$ be a feasible, s-AKKT regular point and assume that $(*)$ holds. Then GCQ is valid at $\bar{x}$.

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$$\mathcal{T}_{\mathcal{F}}^{w}(\bar{x})^{\circ} = \mathcal{M}(\bar{x}, 0).$$

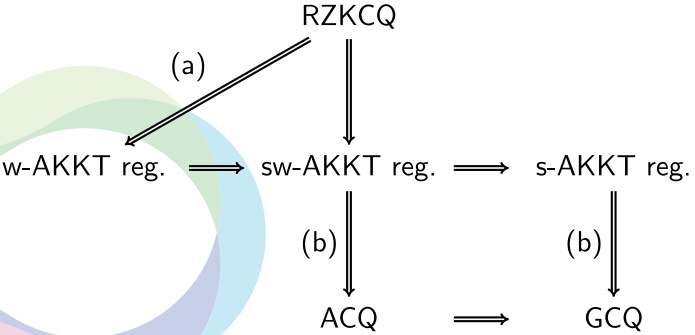
If X is reflexive, GCQ is the weakest CQ, which implies that local minimizers are KKT points (independently of the objective f), [Gould, Tolle, 1975]

Theorem (Börgens, Kanzow, Mehlitz, Wachsmuth (2020))

Let X be refl. and sep. Let $\bar{x} \in \mathcal{F}$ be a feasible, s-AKKT regular point and assume that $(*)$ holds. Then GCQ is valid at $\bar{x}$.

Proof is similar to ACQ

Big picture



Relations between CQs (X assumed to be reflexive).

Relations with labeled arrows only hold under additional assumptions:

(a) requires complete continuity of G and G' as well as $C = X$,

(b) holds whenever X is separable and $(*)$ holds.

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i.e., $C = X$ and $K = \{0\}$.

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Coordinate independent description:

the norm of the Moore–Penrose inverse is bounded

$$A_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}, \qquad A_\varepsilon^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & \delta_\varepsilon \end{pmatrix}, \qquad \delta_\varepsilon = \begin{cases} 0 & \text{if } \varepsilon = 0 \\ \varepsilon^{-1} & \text{else.} \end{cases}$$

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Replacement for Moore–Penrose inverse:

Definition (Kato, 1966)

Let X, Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$ be bounded and linear. The *reduced minimum modulus* of T is defined via

$$\gamma(T) := \inf\{\|T x\|_Y \mid x \in X,\ \operatorname{dist}(x, \ker T) = 1\}.$$

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Generalizes a recent result by Blot, 2018

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Box constraints in Lebesgue spaces

On a domain $\Omega \subset \mathbb{R}^d$, we consider the box constraints

$$u_a \le u \le u_b$$

with $u_a, u_b \in L^2(\Omega)$

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It is well known that RZKCQ is violated, but one can check that sw-AKKT regularity is satisfied

Summary

- generalize AKKT conditions and AKKT regularity from finite to infinite dimensions
- results concerning AKKT points and AKKT regularity
- relations to standard CQs

Outlook

- Application to mathematical problems with complementarity constraints (MPCCs) in infinite dimensions
- Constant rank CQ for inequalities in infinite dimensions?

Questions?