Subdifferentials and Lipschitz properties of translation invariant functionals and applications

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1. Scalarization approach in vector optimization

Let Y be a real linear (topological) space, $A \subseteq Y$, $A \neq \emptyset$, $\mathbb{R}_+$ be the set of all nonnegative numbers. A is said to be solid if int $A \neq \emptyset$, proper if $A \neq \emptyset$ and $A \neq Y$, pointed if $A \cap (-A) \subseteq \{\mathbf{0}\}$, a cone if $\forall a \in A$, $\forall t \in \mathbb{R}_+ :\ ta \in A$, and a convex cone if A is a cone and $A + A = A$. A proper cone is called nontrivial.

$C^+ := \{y^* \in Y^* \mid \forall y \in C :\ y^*(y) \geq 0\}$.

The recession cone of A is defined by

$$A_\infty := \{y \in Y \mid \forall a \in A,\ \forall t \in \mathbb{R}_+ :\ a + ty \in A\}.$$

bar A denotes the domain of the support function of A or the barrier cone of A.

Let Y be equipped with a binary relation generated by a domination set $\Theta \subset Y$ being proper and $0 \in \mathrm{cl}\,\Theta$. Denoting the relation on Y with respect to Θ by $\leq_\Theta$, we have:

$$y_1 \leq_\Theta y_2 :\Longleftrightarrow y_1 \in y_2 - \Theta. \tag{1}$$

When $\Theta = C$ is a nontrivial, closed, convex and pointed cone, $\leq_C$ is a partial order in Y. We do not impose either the convexity property or the conical property for the domination set Θ.

We say that a point $\overline{y} \in \Xi$, where $\Xi \subseteq Y$ is a nonempty subset in Y, is a Θ-minimal point of Ξ with respect to the domination set Θ, if

$$\Xi \cap (\overline{y} - \Theta) = \{\overline{y}\}; \tag{2}$$

i.e., for every $y \in \Xi$, $y \leq_\Theta \overline{y}$ implies $y = \overline{y}$.

Definition 1 (scalarization directions of sets). Let A be a proper subset in a linear space Y. A nonzero vector $\mathbf{k} \in Y$ is called a scalarization direction of A if A does not contain lines parallel to $\mathbf{k}$ and the scalarization condition

$$\forall t \in \mathbb{R}_+ : \; A + t\mathbf{k} \subseteq A \tag{3}$$

holds. Set of all scalarization directions of A: $\text{dir}\,(A)$.

Definition 2 (translation invariant functionals / nonlinear scalarization functionals). Let A be a proper set in a linear space Y and $\mathbf{k} \in \text{dir}\,(A)$ be a scalarization direction of A. The functional $\varphi_{A,\mathbf{k}} : Y \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\varphi_{A,\mathbf{k}}(y) := \inf\{t \in \mathbb{R} \mid y \in t\mathbf{k} - A\}, \tag{4}$$

where $\inf(\emptyset) = +\infty$, is called nonlinear scalarization functional with respect to the set A and the scalarization direction $\mathbf{k}$.

Lemma 3 (GTZ1999, GRTZ2003, Theorem 2.3.1) *Assume that A is a closed set in a topological linear space Y. Then,*

(a) $\varphi_{A,\mathbf{k}}$ *is lower semicontinuous over its domain* dom $(\varphi_{A,\mathbf{k}}) = \mathbb{R}\mathbf{k} - A$*. For every $\tau \in \mathbb{R}$, the τ-level set of $\varphi_{A,\mathbf{k}}$ is given by*

$$\text{Lev}\,(\tau; \varphi_{A,\mathbf{k}}) := \{y \in Y \mid \varphi_{A,\mathbf{k}}(y) \leq \tau\} = \tau\mathbf{k} - A.$$

(b) $\varphi_{A,\mathbf{k}}$ *is translation invariant along the scalarization direction $\mathbf{k}$ (linearly shifted along the scalarization direction $\mathbf{k}$):*

$$\forall y \in Y,\ \forall t \in \mathbb{R}:\ \varphi_{A,\mathbf{k}}(y + t\mathbf{k}) = \varphi_{A,\mathbf{k}}(y) + t.$$

(c) $\varphi_{A,\mathbf{k}}$ *is convex if and only if A is convex, and positively homogeneous if and only if A is a cone.*

Remark: Jaschke and Küchler (2001) have shown that each translation invariant functional has a representation (4).

Remark 1 (on the closedness of A). We always have

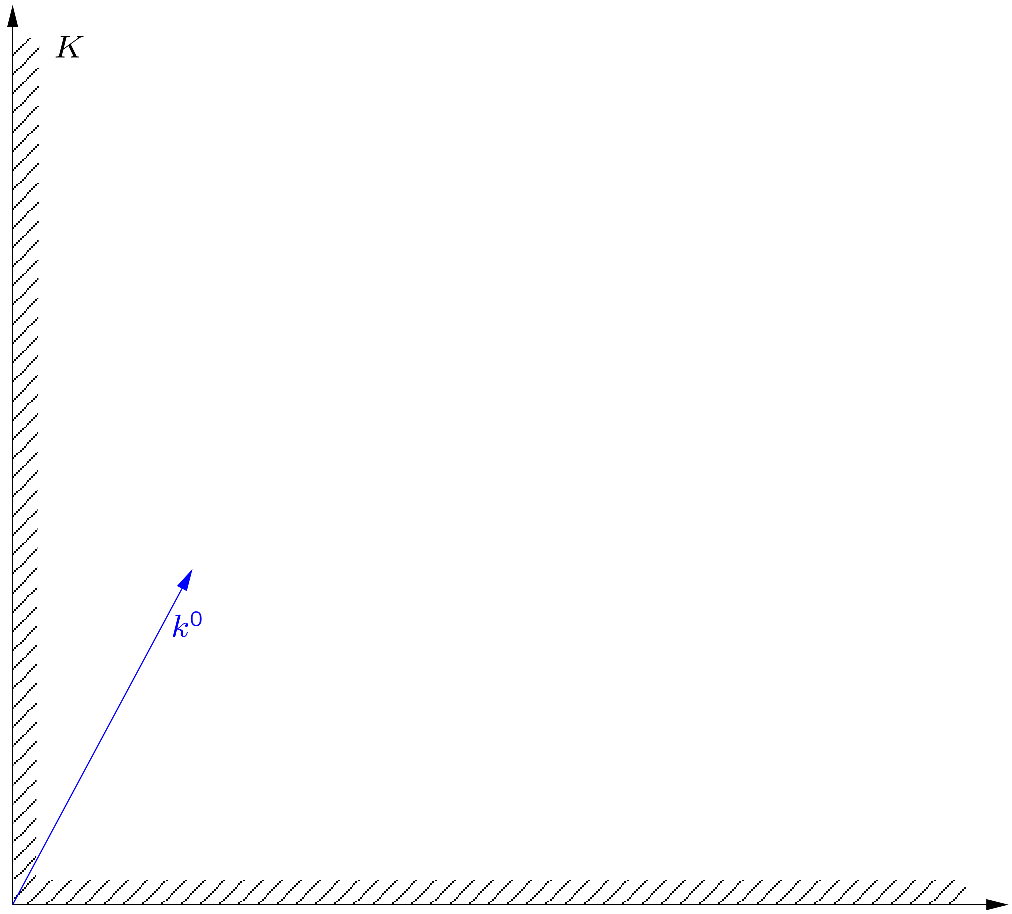
$$\varphi_{A,\mathbf{k}} = \varphi_{\mathrm{vcl}_{\mathbf{k}}(A),\mathbf{k}}$$

for any set A in Y, where $\mathrm{vcl}_{\mathbf{k}}(A)$ is defined by

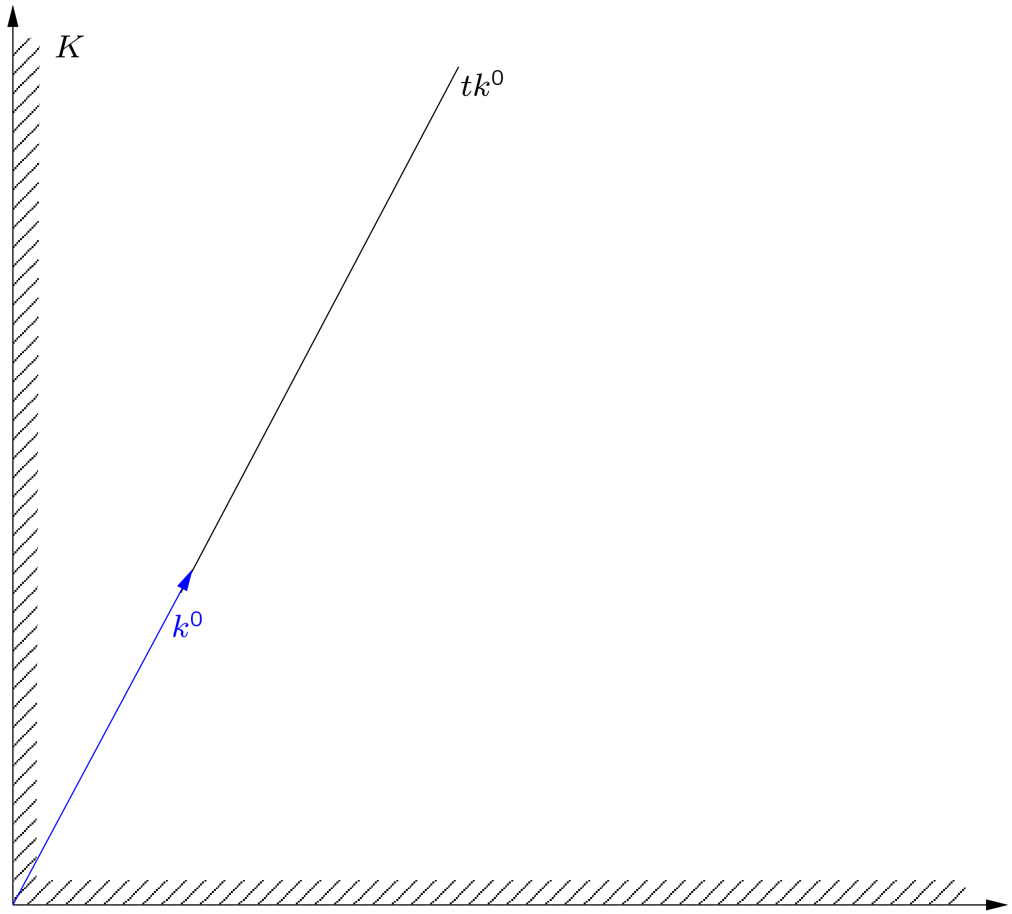
$$\mathrm{vcl}_{\mathbf{k}}(A) := \{y \in Y \mid \forall \tau > 0,\ \exists t \in [0, \tau] :\ y + t\mathbf{k} \in A\}$$

and is called the vector closure of A in the direction $\mathbf{k}$. If A is a closed set, then $A = \mathrm{cl}\,(A) = \mathrm{vcl}_{\mathbf{k}}(A)$. However, the reverse is not true.

Example: Let $A := \{(a, b) \in \mathbb{R}^2 \mid \forall a \in (0, -1) :\ b = -\sqrt{1 - a^2}\}$ be a set and $\mathbf{k} = (1, 1)$ be a direction in $\mathbb{R}^2$. We have $A = \mathrm{vcl}_{\mathbf{k}}(A)$ but $\mathrm{cl}\, A = A \cup \{(-1, 0), (0, -1)\}$.



$$\varphi_{A,k} : Y \to \overline{\mathbb{R}},\ \varphi_{A,k}(y) := \inf\{t \in \mathbb{R} \mid y \in tk - K\},\ (A = K = \mathbb{R}^2_+, k = k^0).$$



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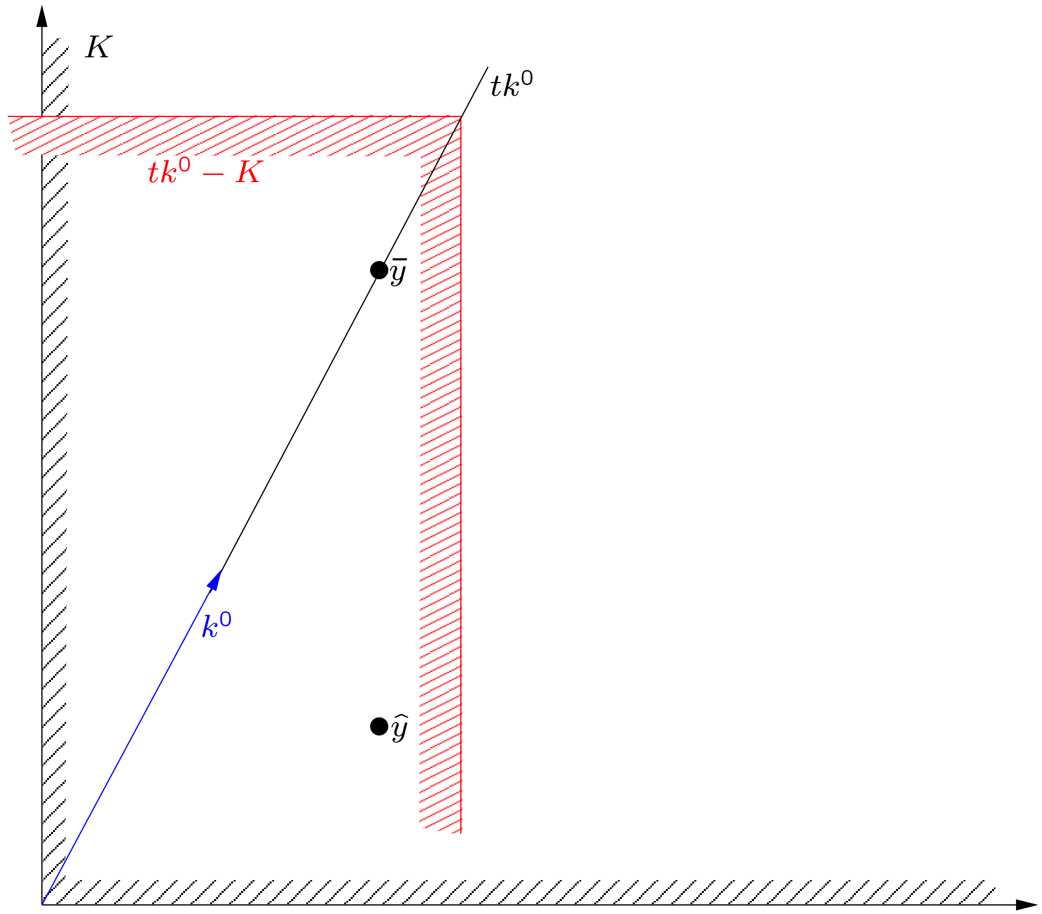

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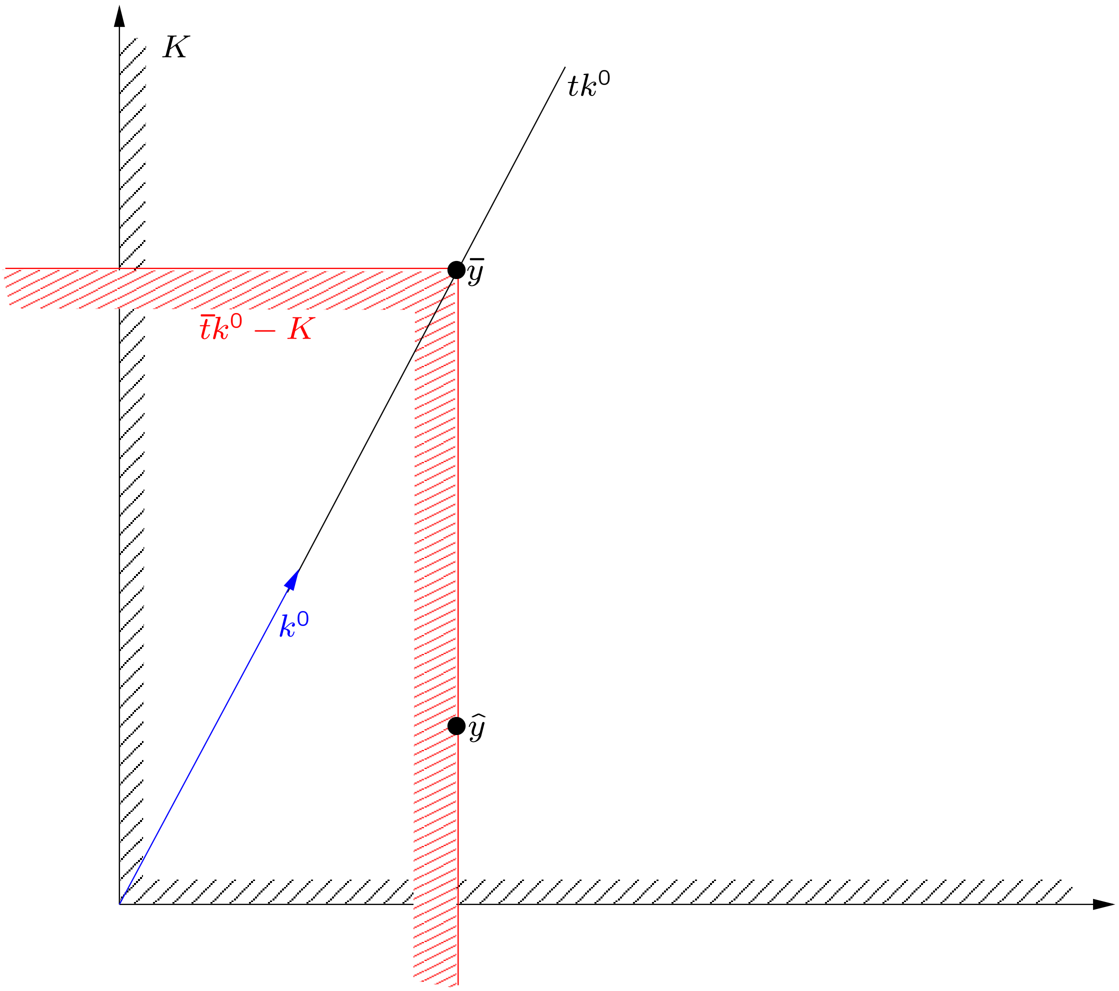

$$\varphi_{A,k}(y) := \inf\{t \in \mathbb{R} \mid y \in tk - K\}, \; \varphi_{A,k}(\bar{y}) = \varphi_{A,k}(\hat{y}) = \bar{t}.$$

Operator Theory: Krasnosel'ski (1964), Rubinov (1977).
Separation theorems, vector optimization: Gerstewitz (Tammer) (1983, 1984), Pascoletti, Serafini (1984), Gerstewitz, Iwanow (1985), Göpfert, Ta., Zălinescu (1999).
In Economics: Luenberger (1992): Shortage function associated to the production possibility set $\mathcal{Y} \subset \mathbb{R}^m$ and $g \in \mathbb{R}^m_+ \setminus \{0\}$:

$$\sigma(g; y) := \inf\{\xi \in \mathbb{R} \mid y - \xi g \in \mathcal{Y}\},$$

Luenberger (1992): Benefit function.
Concepts of robustness in optimization under uncertainty.
Mathematical Finance: Coherent risk measures associated to the set of random variables corresponding to acceptable investments. Artzner et al (1999).
Functional Analysis: Rubinov, Singer (2001) *topical functionals*.

Lemma 4 (DT2009, Lemma 2.1). *Let Y be a topological linear space, C be a nontrivial, closed, solid and convex cone in Y, and $\mathbf{k} \in \operatorname{int}(C)$ be a scalarization direction of C. Then, $\varphi_{C,\mathbf{k}}$ defined in Definition 2 is continuous, sublinear, $\operatorname{int}(C)$-monotone, and translation invariant along the direction $\mathbf{k}$. For every $\overline{y} \in Y$, the subdifferential of $\varphi_{C,\mathbf{k}}$ at $\overline{y} \in Y$ is given by*

$$\partial\varphi_{C,\mathbf{k}}(\overline{y}) = \{y^* \in C^+ \mid y^*(\mathbf{k}) = 1 \wedge y^*(\overline{y}) = \varphi_{C,\mathbf{k}}(\overline{y})\Big\}. \qquad (5)$$

When $\overline{y} = \mathbf{0}$, (5) becomes

$$\partial\varphi_{C,\mathbf{k}}(\mathbf{0}) = C^+ \cap H_1(\mathbf{k}) \text{ with } H_1(\mathbf{k}) := \{y^* \in Y^* \mid y^*(\mathbf{k}) = 1\}.$$

Lemma 5 (DT2009, Theorem 2.2) *Let Y be a real topological linear space, and let A be a nontrivial, closed and convex set in Y, and $\mathbf{k}$ be a scalarization direction of A. Then, for every $\overline{y} \in \operatorname{dom} \varphi_{A,\mathbf{k}}$, the subdifferential (of convex analysis) of $\varphi_{A,\mathbf{k}}$ at $\overline{y}$ is given by*

$$\partial\varphi_{A,\mathbf{k}}(\overline{y}) = \{y^* \in Y^* \mid y^*(\mathbf{k}) = 1 \wedge \forall y \in A:\ y^*(\overline{y} - y) - \varphi_{A,\mathbf{k}}(\overline{y}) \geq 0\}.$$

Lemma 6 (TZ2010, Corollary 4.2) *Let Y be a separated locally convex vector space, $A \subseteq Y$ be a nontrivial, closed and convex set enjoying the free-disposal property, and $\mathbf{k} \notin -A_\infty$. Then, for every $\overline{y} \in Y$ the subdifferential (of convex analysis) of $\varphi_{A,\mathbf{k}}$ at $\overline{y}$ is given by*

$$\partial\varphi_{A,\mathbf{k}}(\overline{y}) = \{y^* \in \operatorname{bar} A \mid y^*(\mathbf{k}) = 1 \wedge \forall y \in A : y^*(\overline{y} - y) - \varphi_{A,\mathbf{k}}(\overline{y}) \geq 0\}.$$

2. Limiting generalized differentiation

Definition 7 (Normal cones) *Let* $\Omega \subseteq X$, $\Omega \neq \emptyset$, X *is an Asplund space.*

(i) *The regular (Fréchet) normal cone to* Ω *at* $x \in \Omega$*:*

$$\widehat{N}(x;\Omega) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{\Omega} x} \frac{x^*(u-x)}{\|u-x\|} \leq 0 \right\}. \qquad (6)$$

(ii) *Assume that* Ω *is locally closed around* $\overline{x} \in \Omega$*. The limiting normal cone (basic normal cone) to* Ω *at* $\overline{x}$*:*

$$\begin{aligned} N(\overline{x};\Omega) &:= \operatorname*{Lim\,sup}_{x \to \overline{x}} \widehat{N}(x;\Omega) \\ &= \left\{ x^* \in X^* \middle| \exists\ x_k \to \overline{x},\ x_k^* \xrightarrow{w^*} x^*,\ x_k^* \in \widehat{N}(x_k;\Omega) \right\}, \end{aligned}$$

where Lim sup *stands for the sequential Painlevé-Kuratowski outer limit as* x *tends to* $\overline{x}$*.*

Definition 8 Let $F : X \rightrightarrows Y$, X, Y Asplund spaces, gph F locally closed around $(\bar{x}, \bar{y}) \in \text{gph}\, F$.

(i) Regular coderivative $\widehat{D}^* F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ of F at $(\bar{x}, \bar{y})$:

$$\widehat{D}^* F(\bar{x}, \bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in \widehat{N}\Big((\bar{x}, \bar{y}); \text{gph}\, F\Big)\}.$$

(ii) Normal limiting coderivative $D_N^* F(\bar{x}, \bar{y}) : Y^* \rightrightarrows X^*$ of F at $(\bar{x}, \bar{y})$

$$D_N^* F(\bar{x}, \bar{y})(y^*) := \Big\{x^* \in X^* \mid (x^*, -y^*) \in N\Big((\bar{x}, \bar{y}); \text{gph}\, F\Big)\}$$

$$= \Big\{x^* \in X^* \mid \exists\, (x_k, y_k) \xrightarrow{\text{gph}\, F} (\bar{x}, \bar{y}),\ (x_k^*, y_k^*) \xrightarrow{w^*} (x^*, y^*)$$

$$\text{with}\ \ (x_k^*, -y_k^*) \in \widehat{N}\Big((x_k, y_k); \text{gph}\, F\Big)\Big\}.$$

(iii) The mixed Mordukhovich/limiting coderivative $D^*_M F(\overline{x}, \overline{y})$: $Y^* \rightrightarrows X^*$ is defined by replacing the weak* convergence $y^*_k \overset{w^*}{\to} y^*$ in (ii) with the norm convergence $y^*_k \xrightarrow{\|\cdot\|} y^*$, i.e.,

$$D^*_M F(\overline{x}, \overline{y})(y^*) := \Big\{ x^* \in X^* \mid \exists\ (x_k, y_k) \xrightarrow{\text{gph}\, F} (\overline{x}, \overline{y}),\ x^*_k \overset{w^*}{\to} x^*,$$

$$y^*_k \xrightarrow{\|\cdot\|} y^*; \text{with}\ \ (x^*_k, -y^*_k) \in \widehat{N}\Big((x_k, y_k); \text{gph}\, F\Big) \Big\}.$$

Definition 9 *Given a set $\Omega \subseteq X \times Y$ in the product of Asplund spaces; in particular,* $\Omega = \text{gph}\, F$, *where* $F : X \to Y$ *is a set-valued mapping. Assume that Ω is locally closed around $(\bar{x}, \bar{y}) \in \Omega$. Ω is sequentially normally compact (SNC) at $\bar{v} = (\bar{x}, \bar{y})$ if for any sequences $\{v_k, x_k^*, y_k^*\}$ satisfying*

$$v_k \xrightarrow{\Omega} \bar{v},\ (x_k^*, y_k^*) \in \widehat{N}(v_k; \Omega)\ (k \in \mathbb{N}), \tag{7}$$

one has

$$(x_k^*, y_k^*) \xrightarrow{w^*} \mathbf{0} \Longrightarrow (x_k^*, y_k^*) \xrightarrow{\|\cdot\|} \mathbf{0}.$$

Assume in addition that the space Y is equipped with a domination set Θ of Y. Then, the epigraph of F with respect to Θ is defined by

$$\operatorname{epi} F := \{(x, y) \in X \times Y \mid y \in F(x) + \Theta\};$$

we omit Θ in the epigraph notation for simplicity. We call the set-valued mapping $\mathcal{E}_F : X \rightrightarrows Y$ defined by

$$\mathcal{E}_F(x) := F(x) + \Theta \tag{8}$$

the epigraphical multifunction with F (and Θ) due to the fact that $\operatorname{gph} \mathcal{E}_F = \operatorname{epi} F$. Adopting coderivatives of set-valued mappings to epigraphical multifunctions, we define subdifferential constructions for F.

Definition 10 (Subdifferentials of set-valued mappings). *Let* $F : X \rightrightarrows Y$ *be a set-valued mapping and* Θ *be a domination set of* Y*. Assume that* epi F *is locally closed at* $(\overline{x}, \overline{y}) \in$ epi F*.*

(i) *The regular subdifferential* $\widehat{\partial} F(\overline{x}, \overline{y}) : Y^* \rightrightarrows X^*$ *of* F *at* $(\overline{x}, \overline{y})$ *is defined by*

$$\widehat{\partial} F(\overline{x}, \overline{y})(y^*) := \widehat{D}^* \mathcal{E}_F(\overline{x}, \overline{y})(y^*).$$

(ii) *The basic subdifferential* $\partial F(\overline{x}, \overline{y}) : Y^* \rightrightarrows X^*$ *of* F *at* $(\overline{x}, \overline{y})$ *is defined by*

$$\partial F(\overline{x}, \overline{y})(y^*) := D_N^* \mathcal{E}_F(\overline{x}, \overline{y})(y^*). \tag{9}$$

(iii) *The singular subdifferential* $\partial^\infty F(\overline{x}, \overline{y})$ *of* F *at* $(\overline{x}, \overline{y})$ *is defined by*

$$\partial^\infty F(\overline{x}, \overline{y}) := D_M^* \mathcal{E}_F(\overline{x}, \overline{y})(\mathbf{0}). \tag{10}$$

3. Subdifferentials of scalarization functionals

Theorem 11 (Subdifferentials of scalarization functionals). Let Y be an Asplund space, A a proper and closed set in Y, and $\mathbf{k} \in \operatorname{dir}(A)$ be a scalarization direction of A. Consider $\varphi_{A,\mathbf{k}}$ and $\overline{y} \in \operatorname{dom} \varphi_{A,\mathbf{k}}$. Then, the regular and limiting subdifferentials of $\varphi_{A,\mathbf{k}}$ at $(\overline{y}, \overline{t}) \in \operatorname{epi} \varphi_{A,\mathbf{k}}$ are (for $\lambda \in \mathbb{R}$)

$$\partial_\bullet \varphi_{A,\mathbf{k}}(\overline{y})(\lambda) = H_\lambda(\mathbf{k}) \cap \Big(- N_\bullet(\overline{t}\mathbf{k} - \overline{y}; A)\Big), \tag{11}$$

where $H_\lambda(\mathbf{k}) := \{y^* \in Y^* \mid y^*(\mathbf{k}) = \lambda\}$, and $\bullet$ stands for both regular and limiting constructions.

Sketch of the proof:

- Define a set-valued mapping $F : \mathbb{R} \rightrightarrows Y$ by $F(t) = t\mathbf{k} - A$.

- Application of Lemma 3 yields $\operatorname{gph} F^{-1} = \operatorname{epi} \varphi_{A,\mathbf{k}}$.

- Apply the coderivative sum rule with equality from Theorem 1.62 in Mordukhovich (2006).

- The normal cone to Cartesian sets is the product of the normal cones to component sets (Proposition 1.2 in Mordukhovich (2006)).

Corollary 12 (Subdifferentials of scalarization functionals). Let Y, A, $\mathbf{k}$, $\varphi_{A,\mathbf{k}}$, $(\overline{y}, \overline{t})$ as in Theorem 11. Then:

(i) The basic subdifferential of $\varphi_{A,\mathbf{k}}$ at $\overline{y}$ is

$$\partial\varphi_{A,\mathbf{k}}(\overline{y}) = H_1(\mathbf{k}) \cap \Big(-N(\varphi_{A,\mathbf{k}}(\overline{y})\mathbf{k} - \overline{y}; A)\Big), \tag{12}$$

where $H_1(\mathbf{k}) := \{y^* \in Y^* \mid y^*(\mathbf{k}) = 1\}$.

(ii) The singular subdifferential of $\varphi_{A,\mathbf{k}}$ at $\overline{y}$ is

$$\partial^\infty\varphi_{A,\mathbf{k}}(\overline{y}) = H_0(\mathbf{k}) \cap \Big(-N(\varphi_{A,\mathbf{k}}(\overline{y})\mathbf{k} - \overline{y}; A)\Big), \tag{13}$$

where $H_0(\mathbf{k}) := \{y^* \in Y^* \mid y^*(\mathbf{k}) = 0\}$.

Definition 13 (Sequential normal epi-compactness of functionals). Let $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ be finite at $\overline{x}$. We say that φ is sequential normal epi-compact (SNEC) at $\overline{x}$ if its epigraph is SNC at $(\overline{x}, \varphi(\overline{x}))$.

Remark: Theorem 11 provides subdifferentials of the scalarization functionals at pairs in the epigraph which are essential for us to study the equivalence between the SNC property of the set A and the SNEC property of the scalarization functional $\varphi_{A,\mathbf{k}}$. Since we do not assume that A is solid, we need to compute the singular subdifferential $\partial^{\infty}\varphi_{A,\mathbf{k}}$ of $\varphi_{A,\mathbf{k}}$ in order to verify the fulfillment of the so-called qualification condition of calculus rules for generalized differentiation.

Corollary 14 (SNEC property of scalarization functionals). Let Y, A, $\mathbf{k}$, $\varphi_{A,\mathbf{k}}$, $(\overline{y},\bar{t})$ as in Theorem 11. $\varphi_{A,\mathbf{k}}$ is SNEC at $\overline{y} \in \operatorname{dom} \varphi_{A,\mathbf{k}}$ if and only if A is SNC at $\varphi_{A,k}(\overline{y})\mathbf{k} - \overline{y}$.

Theorem 15 *Let Y, A, $\mathbf{k}$, $\varphi_{A,\mathbf{k}}$, $(\overline{y},\bar{t})$ as in Theorem 11. Then, $\varphi_{A,\mathbf{k}}$ is locally Lipschitz continuous at $\overline{y} \in \operatorname{dom} \varphi_{A,\mathbf{k}}$ if and only if A is SNC at $\overline{v} := \varphi_{A,\mathbf{k}}(\overline{y})\mathbf{k} - \overline{y}$ and*

$$H_0(\mathbf{k}) \cap (-N(\overline{v}; A)) = \{\mathbf{0}\}, \tag{14}$$

where $H_0(\mathbf{k}) = \{y^* \in Y^* \mid y^*(\mathbf{k}) = 0\}$.

Proof: By Corollary 12, $\partial^\infty \varphi_{A,\mathbf{k}}(\overline{y}) = H_0(\mathbf{k}) \cap (-N(\overline{v}; A))$. By Corollary 14, $\varphi_{A,\mathbf{k}}$ is SNEC at $\overline{y}$. An extended-real-valued function is locally Lipschitz continuous at a point $\overline{y}$ in its domain if and only if its singular sudifferential is trivial and it is SNEC at that point (Theorem 4.10 in Mordukhovich (2006)).

Corollary 16 (Lipschitz continuity of scalarization functionals). *Let Y be an Asplund space, C be a nontrivial, closed, convex and solid cone in Y, and $\mathbf{k} \in \text{int}(C)$ be a scalarization direction of C. Then, $\varphi_{C,\mathbf{k}}$ is locally Lipschitz.*

Corollary 17 (Subdifferentials along scalarization directions). *Let Y, A, $\mathbf{k}$, $\varphi_{A,\mathbf{k}}$, $(\overline{y}, \bar{t})$ as in Theorem 11. Then, for any $\overline{y} \in \text{dom}\, \varphi_{A,\mathbf{k}} = \mathbb{R}\mathbf{k} - A$ and for any $\bar{t} \in \mathbb{R}$, we have*
$\partial\varphi_{A,\mathbf{k}}(\overline{y} + \bar{t}\mathbf{k}) = \partial\varphi_{A,\mathbf{k}}(\overline{y})$ *and* $\partial^{\infty}\varphi_{A,\mathbf{k}}(\overline{y} + \bar{t}\mathbf{k}) = \partial^{\infty}\varphi_{A,\mathbf{k}}(\overline{y})$.

4. Application to set-valued optimization

$$\Theta - \text{Minimize} \quad F(x) \quad \text{subject to} \quad x \in \Omega, \qquad \text{(SP)}$$

where $F : X \rightrightarrows Y$, X, Y are Asplund spaces, $\Omega \neq \emptyset$, $\Omega \subseteq X$, Θ is a domination set in Y with $\text{dir}\,(\Theta) \neq \emptyset$.

Definition 18 (Θ-minimality). *Consider problem* (SP)*. Let* $\overline{x} \in \Omega$ *and* $(\overline{x}, \overline{y}) \in \text{gph}\, F$*. We say that the pair* $(\overline{x}, \overline{y}) \in \text{gph}\, F$ *is a Θ-minimal solution of problem* (SP) *if* $F(\Omega) \cap (\overline{y} - \Theta) = \{\overline{y}\}$.

(NQC($\{F,\Omega\}$)) (norm-convergence qualification condition for $\{F,\Omega\}$):
For any sequence $(x_{1k}, x_{2k}, y_{1k}, x^*_{1k}, x^*_{2k}, y^*_{1k})$ satisfying

$$\left[\begin{array}{l} (x_{1k}, y_{1k}) \in \operatorname{gph} F, x_{2k} \in \Omega, x^*_{1k} \in \widehat{D}^* F(x_{1k}, y_{1k})(y^*_{1k}), \\ x^*_{2k} \in \widehat{N}(x_{2k}; \Omega), (x_{1k}, y_{1k}) \to (\overline{x}, \overline{y}), x_{2k} \to \overline{x}, (x^*_{1k}, x^*_{2k}) \xrightarrow{w^*} (x^*_1, x^*_2) \end{array}\right]$$

one has

$$(\|x^*_{1k} + x^*_{2k}\| \to 0 \ \wedge \ \|y^*_k\| \to 0) \Rightarrow \|x^*_{1k}\| + \|x^*_{2k}\| \to 0.$$

(MQC($\{F,\Theta\}$)) (mixed qualification condition for $\{F,\Theta\}$):

$$\left[y^* \in D^*_M F^{-1}(\overline{y}, \overline{x})(\mathbf{0}) \cap (-N(\mathbf{0}; \Theta)) \text{ and } y^*(\mathbf{k}) = 0\right] \Rightarrow y^* = \mathbf{0}.$$

Remark: (MQC($\{F,\Theta\}$)) could be replaced by the qualification condition for $\{F, \varphi_{\Theta-\overline{y},\mathbf{k}}\}$

$$D^*_M F^{-1}(\overline{y}, \overline{x})(\mathbf{0}) \cap (-\partial^\infty \varphi_{\Theta-\overline{y},\mathbf{k}})(\overline{y})) = \{\mathbf{0}\}.$$

Theorem 19 (necessary conditions for Θ-minimal solutions). *Consider problem (SP) and a Θ-minimal solution $(\overline{x}, \overline{y})$. Let $\mathbf{k} \in \operatorname{dir}(\Theta)$ be a scalarization direction of Θ, and $\varphi = \varphi_{\Theta - \overline{y}, k}$. Assume that the following conditions hold:*

(H1) *(closedness condition) the domination set Θ is locally closed around the origin, $\operatorname{gph} F$ is closed around $(\overline{x}, \overline{y})$, and Ω is locally closed around $\overline{x}$.*

(H2) *(SNC conditions) One of the following conditions holds:*

(a) *Θ is SNC at $\mathbf{0}$ and Ω is SNC at $\overline{x}$;*

(b) *F is SNC at $(\overline{x}, \overline{y})$.*

(H3) *(Qualification conditions)*

Either (NQC($\{F, \Omega\}$)) for $\{F, \Omega\}$ is satisfied in the case of the SNC condition (a), or (MQC($\{F, \Theta\}$)) for $\{F, \Theta\}$ is fulfilled in the case of the SNC condition (b).

The qualification condition for $\{\Theta, F, \Omega\}$ *is satisfied: For any sequence*

$$\{(x_{1k}, x_{2k}, y_{1k}, x^*_{1k}, x^*_{2k}, y^*_{1k})\}$$

satisfying

$$\left[\begin{array}{l} (x_{1k}, y_{1k}) \in \operatorname{gph} F, x_{2k} \in \Omega, x^*_{1k} \in \widehat{D}^* F(x_{1k}, y_{1k})(y^*_{1k}), \\ x^*_{2k} \in \widehat{N}(x_{2k}; \Omega), (x_{1k}, y_{1k}) \to (\overline{x}, \overline{y}), x_{2k} \to \overline{x}, (x^*_{1k}, x^*_{2k}) \xrightarrow{w^*} (x^*_1, x^*_2), \end{array} \right]$$

one has

$$\left[\begin{array}{c} x^*_{1k} \xrightarrow{w^*} x^*_1, x^*_{2k} \xrightarrow{w^*} x^*_2, \|x^*_{1k} + x^*_{2k}\| \to 0, \\ y^*_{1k} \xrightarrow{w^*} -y^*_1, y^*_1 \in -N(\mathbf{0}; \Theta) \cap H_0(\mathbf{k}) \end{array} \right] \Rightarrow \left[\begin{array}{c} x^*_1 = x^*_2 = \mathbf{0} \\ y^*_1 = \mathbf{0} \end{array} \right].$$

Then, there is $y^* \in -N(\mathbf{0}; \Theta)$ *with* $y^*(\mathbf{k}) = 1$ *satisfying*

$$\mathbf{0} \in \partial F(\overline{x}, \overline{z})(\overline{y}^*) + N(\overline{x}; \Omega). \tag{15}$$

5. Application in approximation theory

X, Y and Z are real Banach spaces, Θ is a nontrivial, closed, convex and pointed cone in Y.

Vector-valued norm: $|||\cdot||| : Z \to \Theta$ which for all $z, z_1, z_2 \in Z$ and for all $\lambda \in \mathbb{R}$ satisfies:

(1) $|||z||| = 0 \Longleftrightarrow z = 0$;

(2) $|||\lambda\ z\ ||| = |\ \lambda\ |\ \ |||z|||$;

(3) $|||z_1 + z_2|||\ \in\ |||z_1||| + |||z_2||| -\ \Theta$.

Subdifferential (denoted $\partial^{\leq}$) for the vector-valued norm $|||\cdot|||$ ($L(Z, Y)$ denotes the space of linear continuous operators from Z into Y):

$$\partial^{\leq}|||\cdot|||(z_0) = \{T \in L(Z, Y) | T(z_0) = |||z_0||| \wedge (\forall z \in Z : |||z||| - T(z) \in \Theta)\}.$$

Furthermore, we assume that $\partial^{\leq}\|\cdot\| \neq \emptyset$.

Suppose that the cost function $g : X \times W \to Y$ is locally Lipschitz, $\Omega \subseteq X$, $A_i \in L(X, Z)$, $a^i \in Z$ and $\alpha_i \geq 0$ $(i = 1, \ldots, n)$. Consider the vector-valued approximation problem (VOP)

$$\text{minimize } f(x, w) := g(x, w) + \sum_{i=1}^{n} \alpha_i \|A_i(x) - a^i\| \text{ subject to } x \in \Omega,$$

where $w \in W(x)$ stands for the control parameter and minimization is understood with respect to the partial order generated by a proper, closed, pointed and convex cone $\Theta \subseteq Z$ in (1) and $\Omega \subseteq X$ is closed. By considering $G(x) := \{g(x, w) | w \in W(x)\}$, (VOP) is equivalent to the set-valued approximation problem (SP):

$$\text{minimize } F(x) := G(x) + \{\sum_{i=1}^{n} \alpha_i \|A_i(x) - a^i\|\} \text{ subject to } x \in \Omega.$$

Theorem 20 Suppose that X, Y, Z are reflexive Banach spaces, $\Theta \subseteq Z$ a proper convex Daniell cone with a weakly compact base, $\|\!| \cdot |\!\|$ is continuous and Ω is a closed subset of X. Assume that $(\overline{x}, \overline{y})$ with $\overline{x} \in \Omega$, $\overline{z} \in G(\overline{x})$, and $\overline{y} := \overline{z} + \sum_{i=1}^{n} \alpha_i |\!\|A_i(\overline{x}) - a^i|\!\|$ is a Θ-minimal solution of problem (SP) and that G is Lipschitzian-like at $(\overline{x}, \overline{y})$. Assume that Θ is SNC at $\mathbf{0}$ and the qualification condition $y^* \in -N(\mathbf{0}; \Theta) \cap H_0(\mathbf{k}), \mathbf{0} \in D^*F(\overline{x}, \overline{v})(y^*) + N(\overline{x}; \Omega) \Rightarrow y^* = \mathbf{0}$ for some $\mathbf{k} \in \operatorname{dir}(\Theta)$ holds. Then, $\exists y^* \in Z^*$ with $y^*(\mathbf{k}) = 1$ and

$$0 \in \partial G(\overline{x}, \overline{y})(y^*) + \sum_{i=1}^{n} \alpha_i A_i^* y^* T_i + N(\overline{x}; \Omega),$$

where $T_i \in L(Z, Y)$, $T_i(A_i(\overline{x}) - a^i) = |\!\|A_i(\overline{x}) - a^i|\!\|$,
$\forall z \in Z: \ |\!\|z|\!\| - T_i(z) \in \Theta, \ (i = 1, \ldots, n)$.

6. Conclusions

- To derive optimality conditions for set optimization problems based on other solution concepts (set approach) and to develop numerical procedures based on scalarization.

- Compute the subdifferential of scalarizing functionals related to vector optimization w.r.t. variable domination structure.

- Computation of the subdifferential of the vector-valued norm where a general domination set Θ is involved.

- Applications in locational analysis and optimization under uncertainty.

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