CONTINUOUS SELECTIONS FOR INVERSE MAPPINGS IN BANACH SPACES

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RADEK CIBULKA AND MARIÁN FABIAN

Abstract. Influenced by a recent work by A. V. Arutyunov, A. F. Izmailov, and S. E. Zhukovskiy, we establish a general Ioffe-type criterion guaranteeing the existence of a continuous and calm selection for the inverse of a single-valued uniformly continuous mapping between Banach spaces with a closed domain. We show that the general statement yields elegant proofs following the same pattern as in the case of the usual openness with a linear rate by considering mappings instead of points. As in the case of the Ioffe's criterion for linear openness around the reference point, this allows us to avoid the iteration, that is, the construction of a sequence of continuous functions the limit of which is the desired continuous selection for the inverse mapping, which is illustrated on the proof of the Bartle-Graves theorem. Then we formulate sufficient conditions based on approximations given by positively homogeneous mappings and bunches of linear operators. The talk is based on a joint work with Marián Fabian.

Warning: This file contains necessary *minimal* selection of results, proofs, and references, only! A complete list of references as well as detailed historical comments can be found in our pre-print. Also several statements are slightly simplified and adjusted.

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1. Introduction

The Clarke's inverse function theorem [6, Theorem 1] generalizes the classical inverse mapping theorem from C^1 smooth to locally Lipschitz continuous mappings and can be stated as follows:

If all matrices in the Clarke generalized Jacobian at a reference point of a locally Lipschitz continuous mapping from a Euclidean space into itself are non-singular, then this mapping admits locally an inverse which is Lipschitz continuous.

For mappings with values in another Euclidean space having a smaller dimension, we have the Pourciau's open mapping theorem [16, Theorem 6.1]:

If all matrices in the Clarke generalized Jacobian at a reference point of a locally Lipschitz continuous mapping from a Euclidean space into another Euclidean space have full rank, then this mapping is open around the reference point with a linear rate.

This statement means, in particular, that the inverse (set-valued) mapping admits a local (single-valued) selection which is defined around and calm at the reference point. A. V. Arutyunov, A. F. Izmailov, and S. E. Zhukovskiy [2] proved that one can find a selection which is, in addition, continuous around the reference point.

In the second section, we use the Ekeland's variational principle to establish Ioffe-type conditions guaranteeing the existence of a solution of a non-linear non-smooth equation. This statement easily implies criteria for the usual openness with a linear rate at/around a point for mappings acting in Banach spaces as well as in Fréchet spaces. Most importantly, it also implies a sufficient condition for the existence of a continuous and calm selection for the inverse of a single-valued Lipschitz continuous mapping between Banach spaces having a closed domain. In the third section, by example of the Bartle-Graves theorem, we illustrate that the general criterion yields elegant proofs following the same pattern, as in the case of linear openness, by considering mappings instead of points. In the fourth section, we concentrate on approximations of a mapping in question by positively homogeneous mappings. We derive conditions ensuring the usual openness with a linear rate around the reference point. Then, under stronger assumptions, we present a related statement on the existence of a continuous and calm selection for the inverse mapping. In the last section, we derive corollaries in case that a positively homogeneous approximation of the mapping under consideration is generated by bunches of linear operators, in order to illustrate that our general results cover and unify various statements from the literature.

2. Regularity criteria

The Ekeland's variational principle implies a condition guaranteeing that a non-linear non-smooth equation is solvable for a given right-hand side.

Proposition 2.1. *Let X and Y be non-empty sets, let $\bar{x} \in X$, let a mapping $g : X \to Y$ be defined on all of X, and fix a point $y \in Y$. Suppose that there exists a complete metrics $\zeta = \zeta_y$ on X and a function $\vartheta = \vartheta_y : Y \to [0, \infty]$, defined on the whole Y, such that $\vartheta(g(\bar{x})) < \infty$, that $\vartheta^{-1}(0) = \{y\}$, and that the function $\vartheta \circ g$ is lower semi-continuous on (X, ζ). If for each $x \in X$ satisfying*

$$0 < \vartheta(g(x)) \leq \vartheta(g(\bar{x})) - \zeta(x, \bar{x}) \tag{1}$$

there is a (better) point $\hat{x} \in X$ such that

$$\vartheta(g(\hat{x})) < \vartheta(g(x)) - \zeta(\hat{x}, x), \tag{2}$$

then there exists a point $u \in X$ such that $g(u) = y$ and $\zeta(u, \bar{x}) \leq \vartheta(g(\bar{x}))$.

Proof. If $g(\bar{x}) = y$, then $u := \bar{x}$ satisfies the conclusion. Further, assume that $y \neq g(\bar{x})$. The Ekeland's variational principle yields a point $u \in X$ such that

$$\vartheta(g(u)) \leq \vartheta(g(\bar{x})) - \zeta(u, \bar{x}) \tag{3}$$

and

$$\vartheta(g(x)) \geq \vartheta(g(u)) - \zeta(x, u) \quad \text{for every} \quad x \in X. \tag{4}$$

Then $\zeta(u, \bar{x}) \leq \vartheta(g(\bar{x}))$. Assume that $g(u) \neq y$. Thus $\vartheta(g(u)) > 0$ and (3) says that (1), with $x := u$, holds. By the assumption, there is a point $\hat{x} \in X$ such that $\vartheta(g(\hat{x})) < \vartheta(g(u)) - \zeta(\hat{x}, u)$. Setting $x := \hat{x}$ in (4), we get that $\vartheta(g(\hat{x})) \geq \vartheta(g(u)) - \zeta(\hat{x}, u)$, a contradiction. Consequently $g(u) = y$. □

Remark 2.2. If the values of ϑ are finite, then the result above follows from [12, Lemma 1], a slight reformulation and generalization of the Caristi's fixed point principle, proved by an iterative procedure since that time the use of the Ekeland's variational principle was not so common in variational analysis. Note that a less general version of [12, Lemma 1] was rediscovered as [1, Theorem 3].

Since the function ϑ as well as the metrics ζ in Proposition 2.1 are allowed to depend on the choice of the right-hand side y, we get [4, Proposition 3] in Fréchet spaces. This is straightforward and a precise formulation requests several specific notions, we leave performing details to an interested reader (if any).

We derive two corollaries of the above result in the Banach space setting (although they are valid in metric spaces as well).

Proposition 2.3. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces, let $\bar{x} \in X$, and let c and r be positive constants. Consider a mapping $f : X \to Y$, with closed domain $\Omega \ni \bar{x}$, which is continuous on $B_X[\bar{x}, r] \cap \Omega$. Assume that for every $x \in B_X[\bar{x}, r] \cap \Omega$ and every $y \in B_Y[f(\bar{x}), cr]$ satisfying*

$$0 < \|f(x) - y\| \leq \|f(\bar{x}) - y\| - c\,\|x - \bar{x}\| \tag{5}$$

there is a (better) point $\hat{x} \in \Omega$ such that

$$\|f(\hat{x}) - y\| < \|f(x) - y\| - c\,\|\hat{x} - x\|.$$

Then $f\big(B_X[\bar{x}, t] \cap \Omega\big) \supset B_Y[f(\bar{x}), ct]$ for every $t \in (0, r]$.

Proof. Fix an arbitrary $t \in (0, r]$ and $y \in B_Y[f(\bar{x}), ct]$. Apply Proposition 2.1 with $X := B_X[\bar{x}, r] \cap \Omega$, $\vartheta(v) := \|v - y\|/c$, $v \in Y$, and $g := f$. □

The proofs in [2] are based on a reformulation of the Caristi's fixed point principle (see Remark 2.2). Proposition 2.1 gives a statement guaranteeing the existence of a selection for the inverse mapping which is defined around, calm at, and `continuous around` the reference point. Given Banach spaces $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$, denote by $\mathcal{C}(D; R)$ the set of all continuous mappings $h : X \to Y$, with domain $D \subset X$ and values in $R \subset Y$.

Proposition 2.4. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces, let $\bar{x} \in X$, and let c and r be positive constants. Consider a mapping $f : X \to Y$, with closed domain $\Omega \ni \bar{x}$, which is Lipschitz or just uniformly continuous on $B_X[\bar{x}, r] \cap \Omega$. Put $\bar{y} := f(\bar{x})$ and assume that for each $\varphi \in \mathcal{C}(B_Y[\bar{y}, cr]; \Omega)$ such that $f(\varphi(y_0)) \neq y_0$ for some $y_0 \in B_Y[\bar{y}, cr]$ and that*

$$(6) \qquad \|f(\varphi(y)) - y\| \leq \|y - \bar{y}\| - c\,\|\varphi(y) - \bar{x}\| \quad \text{for each} \quad y \in B_Y[\bar{y}, cr]$$

there is a mapping $\widehat{\varphi} \in \mathcal{C}(B_Y[\bar{y}, cr]; \Omega)$ such that (6) with $\varphi := \widehat{\varphi}$ holds and that

$$(7) \quad \sup_{y \in B_Y[\bar{y}, cr]} \|f(\widehat{\varphi}(y)) - y\| < \sup_{y \in B_Y[\bar{y}, cr]} \|f(\varphi(y)) - y\| - c \sup_{y \in B_Y[\bar{y}, cr]} \|\widehat{\varphi}(y) - \varphi(y)\|.$$

Then there exists a mapping $\widetilde{\varphi} \in \mathcal{C}(B_Y[\bar{y}, cr]; \Omega)$ such that $f(\widetilde{\varphi}(y)) = y$ and $c\,\|\widetilde{\varphi}(y) - \bar{x}\| \leq \|y - \bar{y}\|$ for each $y \in B_Y[\bar{y}, cr]$.

Proof. Put $\mathcal{X} := \{\varphi \in \mathcal{C}(B_Y[\bar{y}, cr]; \Omega) : \varphi \text{ satisfies (6)}\}$. The set $\mathcal{X}$ is non-empty because it contains the constant function $\overline{\varphi}(\cdot) \equiv \bar{x}$. Put $g(\varphi) := f \circ \varphi,\ \varphi \in \mathcal{X}$, and $\mathcal{Y} := \mathcal{C}(B_Y[\bar{y}, cr]; B_Y[\bar{y}, 2cr])$.

We intend to apply Proposition 2.1. Fix for a while any $\varphi \in \mathcal{X}$. Then $g(\varphi)$ is well-defined and maps into $B_Y[\bar{y}, 2cr]$, because φ maps into $B_X[\bar{x}, r] \cap \Omega$ and for every $y \in B_Y[\bar{y}, cr]$ we have by (6) that

$$\|g(\varphi)(y) - \bar{y}\| = \|f(\varphi(y)) - \bar{y}\| \leq \|f(\varphi(y)) - y\| + \|y - \bar{y}\| \leq 2\|y - \bar{y}\| \leq 2cr.$$

The continuity of f immediately implies that $g(\varphi)$ is continuous. Thus g maps $\mathcal{X}$ into $\mathcal{Y}$. Let $I(y) := y,\ y \in B_Y[\bar{y}, cr]$. Clearly $I \in \mathcal{Y}$. Define $\zeta : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$\zeta(\varphi_1, \varphi_2) := \sup_{y \in B_Y[\bar{y}, cr]} \|\varphi_1(y) - \varphi_2(y)\|, \quad \varphi_1, \varphi_2 \in \mathcal{X}.$$

That ζ is a complete metrics on $\mathcal{X}$ follows from [10, Theorem 4.3.13] and from the easily verifiable fact that the validity of (6) is conserved when going to limits in the metrics ζ. Define a function $\vartheta : \mathcal{Y} \to [0, \infty)$ by

$$\vartheta(\eta) := c^{-1} \sup_{y \in B_Y[\bar{y}, cr]} \|\eta(y) - y\|, \quad \eta \in \mathcal{Y}.$$

Then $\vartheta(g(\overline{\varphi})) = r < \infty$ and $\vartheta^{-1}(0) = I$. The uniform continuity of f at once implies that g is continuous. From this, we can easily conclude that the function $\vartheta \circ g$ is lower semi-continuous on $(\mathcal{X}, \zeta)$.

Now, consider any $\varphi \in \mathcal{X}$ satisfying (1) (where $x := \varphi$ and $\bar{x} := \bar{\varphi}$), that is, $0 < \vartheta(g(\varphi)) \leq \vartheta(g(\bar{\varphi})) - \zeta(\varphi, \bar{\varphi})$. Then, in particular $\vartheta(g(\varphi)) > 0$, i.e., $g(\varphi) \neq I$, i.e., $f(\varphi(y)) \neq y$ for some $y \in B_Y[\bar{y}, cr]$. Also, (6) holds as $\varphi \in \mathcal{X}$. Thus there is a mapping $\widehat{\varphi} \in \mathcal{X}$ such that (7) holds, which is (2), where $x := \varphi$ and $\hat{x} := \widehat{\varphi}$. Applying Proposition 2.1, with $y := I$, we find a mapping $\widetilde{\varphi} \in \mathcal{X}$ such that $g(\widetilde{\varphi}) = I$ and $\zeta(\widetilde{\varphi}, \bar{\varphi}) \leq \vartheta(g(\bar{\varphi}))$. Thus, for every $y \in B_Y[\bar{y}, cr]$, we have $f(\widetilde{\varphi}(y)) = y$ and $c\|\widetilde{\varphi}(y) - \bar{x}\| \leq \|y - \bar{y}\| - \|f(\widetilde{\varphi}(y)) - y\| = \|y - \bar{y}\|$; here we used that $\widetilde{\varphi} \in \mathcal{X}$, and so (6) with $\varphi := \widetilde{\varphi}$ holds. □

3. Approximation by one linear operator

As an illustration, let us prove the (updated) Graves' theorem [9, Theorem 5D.1] and as well as the theorem by Bartle and Graves [9, Theorem 5J.3]. Proposition 2.3 and Proposition 2.4, allow us to avoid the iteration, that is, the construction of a sequence of points [continuous functions] the limit of which is the desired solution of the equation $f(x) = y$ [continuous selection for f^{-1}].

Theorem 3.1. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces, let $\bar{x} \in X$, let α, μ, and δ be positive constants. Consider a mapping $f : X \to Y$ along with a linear bounded mapping $A : X \to Y$ such that $A(B_X) \supset (\alpha + \mu)B_Y$ and that*

$$\|f(u) - f(x) - A(u - x)\| \leq \mu \|u - x\| \quad \text{for each} \quad u, x \in B_X[\bar{x}, \delta]. \tag{8}$$

Then for each $c \in (0, \alpha)$, each $\tilde{x} \in B_X(\bar{x}, \delta)$, and each $r \in (0, \delta - \|\tilde{x} - \bar{x}\|]$, the following statements hold true:

- (*i*) *For an arbitrary $y \in B_Y[f(\tilde{x}), cr]$, there is a point $x \in B_X[\tilde{x}, r]$ such that $f(x) = y$ and $c\|x - \tilde{x}\| \leq \|y - f(\tilde{x})\|$.*
- (*ii*) *There is a function $\widetilde{\varphi} \in \mathcal{C}(B_Y[f(\tilde{x}), cr]; X)$ such that $f(\widetilde{\varphi}(y)) = y$ and $c\|\widetilde{\varphi}(y) - \tilde{x}\| \leq \|y - f(\tilde{x})\|$ for each $y \in B_Y[f(\tilde{x}), cr]$.*

Proof. Fix arbitrary c, $\tilde{x}$, and r as in the premise. Pick an $\varepsilon > 0$ such that $c < \alpha - \varepsilon$. Put $\tilde{y} := f(\tilde{x})$ and $\lambda := \alpha + \mu - \varepsilon$. Pick a $\gamma \in (0, 1/(\alpha + \mu))$.
As $A(B_X) \supset (\alpha + \mu)B_Y$, conclude that

$$\forall\, v \in Y \;\; \exists\, h \in X : \; Ah = v \text{ and } \lambda\|h\| \leq \|v\|. \tag{9}$$

(i) Fix arbitrary $x \in B_X[\tilde{x}, r]$ and $y \in B_Y[\tilde{y}, cr]$ such that

$$0 < \|f(x) - y\| \leq \|\tilde{y} - y\| - c\|x - \tilde{x}\|.$$

Let $v := y - f(x)$. Then $x \in B_X[\tilde{x}, r] \subset B_X[\bar{x}, \delta]$. By (9), there is an $h \in X$ such that $Ah = v$ and $\lambda\|h\| \leq \|v\|$. Set $\hat{x} := x + \gamma\lambda h$. Then

$$\|\hat{x} - x\| = \gamma\lambda\|h\| \leq \gamma\|v\|,$$

and, as $c\gamma < 1$, we conclude that

$$c\|\hat{x} - \tilde{x}\| \leq c\|x - \tilde{x}\| + c\|\hat{x} - x\| \leq c\|x - \tilde{x}\| + c\gamma\|v\| \leq \|\tilde{y} - y\| \leq cr.$$

Therefore $\hat{x} \in B_X[\tilde{x}, r] \subset B_X[\bar{x}, \delta]$. By (8), we get that

$$\|f(\hat{x}) - f(x) - A(\gamma\lambda h)\| \leq \mu\gamma\lambda\|h\| \leq \mu\gamma\|v\|.$$

Since $A(\gamma\lambda h) = \gamma\lambda v$ and $\gamma\lambda < 1$, we may estimate

$$\begin{aligned}
\|y - f(\hat{x})\| &\leq \|y - f(x) - \gamma\lambda v\| + \|f(x) - f(\hat{x}) + A(\gamma\lambda h)\| \\
&= (1 - \gamma\lambda)\|v\| + \|f(\hat{x}) - f(x) - A(\gamma\lambda h)\| \\
&\leq (1 - \gamma\lambda)\|v\| + \mu\gamma\|v\| = (1 - \gamma(\alpha - \varepsilon))\|v\| < (1 - \gamma c)\|v\| \\
&\leq \|v\| - \gamma c\lambda\|h\| = \|y - f(x)\| - c\|\hat{x} - x\|.
\end{aligned}$$

Proposition 2.3, with $\bar{x} := \tilde{x}$ and $t := r$, implies the conclusion in (i).

(ii) In view of (9), the mapping

$$Y \ni v \longmapsto \Xi(v) := \big\{h \in X :\ Ah = v \text{ and } \lambda\|h\| \le \|v\|\big\}$$

has non-empty closed convex values and is lower semi-continuous on Y. Indeed, fix an arbitrary $\bar{v} \in Y$. If $\bar{v} = 0$, then Ξ is continuous at $\bar{v}$ because $\Xi(0) = \{0\}$. Suppose that $\bar{v} \ne 0$. Let Ω be an open set in X such that $\Xi(\bar{v}) \cap \Omega \ne \emptyset$. Pick any $\tilde{h} \in \Xi(\bar{v}) \cap \Omega$. There is a non-zero $\hat{h}$ such that $A\hat{h} = \bar{v}$ and $\lambda\|\hat{h}\| < (\alpha+\mu)\|\hat{h}\| \le \|\bar{v}\|$. As $\tilde{h}$, $\hat{h} \in A^{-1}(\bar{v})$, we can find $\tau \in (0,1)$ such that $\bar{h} := \tilde{h} + \tau(\hat{h} - \tilde{h}) \in \Omega \cap A^{-1}(\bar{v})$. Since $\lambda\|\bar{h}\| < \|\bar{v}\|$, there are open neighborhoods $U \subset \Omega$ of $\bar{h}$ and V of $\bar{v}$ such that $\lambda\|h\| < \|v\|$ whenever $h \in U$ and $v \in V$. As A is an open mapping, the inverse A^{-1} is lower semi-continuous at $\bar{v}$. Hence there is a neighborhood $W \subset V$ of $\bar{v}$ such that $A^{-1}(v) \cap U \ne \emptyset$ for each $v \in W$. Consequently, for each $v \in W$ we have $\Xi(v) \cap \Omega \supset A^{-1}(v) \cap U \ne \emptyset$.

By the Michael's selection theorem, there is a continuous mapping $\eta : Y \to X$ such that

$$(10) \qquad A\eta(v) = v \quad \text{and} \quad \lambda\|\eta(v)\| \le \|v\| \quad \text{for each} \quad v \in Y.$$

We are going to use Proposition 2.4 with $(\bar{x}, \bar{y})$ replaced by $(\tilde{x}, \tilde{y})$. Inequality (8) ensures that f is Lipschitz continuous on $B_X[\bar{x}, \delta] \supset B_X[\tilde{x}, r]$. Let $\varphi \in \mathcal{C}(B_Y[\tilde{y}, cr]; X)$ be such that $f(\varphi(y_0)) \ne y_0$ for some $y_0 \in B_Y[\tilde{y}, cr]$ and that

$$(11) \qquad \|f(\varphi(y)) - y\| \le \|y - \tilde{y}\| - c\,\|\varphi(y) - \tilde{x}\| \quad \text{for each} \quad y \in B_Y[\tilde{y}, cr].$$

We shall construct a $\widehat{\varphi} \in \mathcal{C}(B_Y[\tilde{y}, cr]; X)$ satisfying (11) with $\varphi := \widehat{\varphi}$ such that

$$(12) \quad \sup_{y \in B_Y[\tilde{y}, cr]} \|f(\widehat{\varphi}(y)) - y\| < \sup_{y \in B_Y[\tilde{y}, cr]} \|f(\varphi(y)) - y\| - c \sup_{y \in B_Y[\tilde{y}, cr]} \|\widehat{\varphi}(y) - \varphi(y)\|.$$

Define $v : Y \to Y$ and $\widehat{\varphi} : Y \to X$ for each $y \in B_Y[\tilde{y}, cr]$, respectively, by

$$(13) \qquad v(y) := y - f(\varphi(y)) \quad \text{and} \quad \widehat{\varphi}(y) := \varphi(y) + \gamma\lambda\,\eta(v(y)).$$

Given an arbitrary $y \in B_Y[\tilde{y}, cr]$, (11) implies that

$$\varphi(y) \in B_X[\tilde{x}, r] \subset B_X[\bar{x}, \delta] \quad \text{and} \quad \|v(y)\| = \|y - f(\varphi(y))\| \le \|y - \tilde{y}\| \le cr.$$

Thus $v(\cdot)$ and $\widehat{\varphi}(\cdot)$ are well-defined and continuous. By (10), we get that

$$(14) \qquad \|\widehat{\varphi}(y) - \varphi(y)\| = \gamma\lambda\,\|\eta(v(y))\| \le \gamma\,\|v(y)\| \quad \text{for all} \quad y \in B_Y[\tilde{y}, cr].$$

As $c\gamma < 1$, (11) and (14) imply that for each $y \in B_Y[\tilde{y}, cr]$ we have that

$$\begin{aligned} c\,\|\widehat{\varphi}(y) - \tilde{x}\| &\le c\,\|\varphi(y) - \tilde{x}\| + c\,\|\widehat{\varphi}(y) - \varphi(y)\| \\ (15) \qquad &\le \|y - \tilde{y}\| - \|v(y)\| + c\gamma\,\|v(y)\| \le \|y - \tilde{y}\| \le cr; \end{aligned}$$

thus $\widehat{\varphi}(y) \in B_X[\tilde{x}, r] \subset B_X[\bar{x}, \delta]$.

We `claim` that

$$(16) \qquad \|y - f(\widehat{\varphi}(y))\| \le \|v(y)\|(1 - (\alpha - \varepsilon)\gamma) \quad \text{for each} \quad y \in B_Y[\tilde{y}, cr].$$

To show this, pick an arbitrary $y \in B_Y[\tilde{y}, cr]$. If $y = f(\varphi(y))$ then (16) holds trivially because $v(y) = 0$ and thus $\widehat{\varphi}(y) = \varphi(y)$. Assume further that $y \ne$

$f(\varphi(y))$, i.e. $v(y) \neq 0$. We note that $\eta(v(y)) \neq 0$ because $A\eta(v(y)) = v(y)$. From (8) and (14), we get that

$$\|f(\widehat{\varphi}(y)) - f(\varphi(y)) - A(\widehat{\varphi}(y) - \varphi(y))\| \leq \mu\gamma\|v(y)\|. \tag{17}$$

Since $A(\widehat{\varphi}(y) - \varphi(y)) = \gamma\lambda v(y)$ and $\gamma\lambda < 1$, (17) yields that

$$\begin{aligned} \|y - f(\widehat{\varphi}(y))\| &= \|(1-\gamma\lambda)v(y) + f(\varphi(y)) - f(\widehat{\varphi}(y)) + A(\widehat{\varphi}(y) - \varphi(y))\| \\ &\leq (1-\gamma\lambda)\,\|v(y)\| + \mu\gamma\|v(y)\| = \|v(y)\|(1-(\alpha-\varepsilon)\gamma). \end{aligned}$$

Inequality (16) is proved. As $c < \alpha - \varepsilon$, given any $y \in B_Y[\tilde{y}, cr]$, (16) and (15) imply that

$$\begin{aligned} \|y - f(\widehat{\varphi}(y))\| + c\,\|\widehat{\varphi}(y) - \tilde{x}\| &\leq \|v(y)\| - c\gamma\|v(y)\| + \|y - \tilde{y}\| - \|v(y)\| + c\gamma\|v(y)\| \\ &= \|y - \tilde{y}\|. \end{aligned}$$

Hence (11) holds with $\varphi := \widehat{\varphi}$. By (14),

$$\sup_{y \in B_Y[\tilde{y},cr]} \|\widehat{\varphi}(y) - \varphi(y)\| \leq \sup_{y \in B_Y[\tilde{y},cr]} \gamma\|v(y)\|. \tag{18}$$

By the choice of φ, we have $y_0 \neq f(\varphi(y_0))$, therefore $\sup_{y \in B_Y[\tilde{y},cr]} \|v(y)\| > 0$. As $c < \alpha - \varepsilon$, combining (16) and (18) we obtain

$$\begin{aligned} \sup_{y \in B_Y[\tilde{y},cr]} \|y - f(\widehat{\varphi}(y))\| &< (1-c\gamma) \sup_{y \in B_Y[\tilde{y},cr]} \|v(y)\| \\ &= \sup_{y \in B_Y[\tilde{y},cr]} \|y - f(\varphi(y))\| - c \sup_{y \in B_Y[\tilde{y},cr]} \gamma\|v(y)\| \\ &\leq \sup_{y \in B_Y[\tilde{y},cr]} \|y - f(\varphi(y))\| - c \sup_{y \in B_Y[\tilde{y},cr]} \|\widehat{\varphi}(y) - \varphi(y)\|, \end{aligned}$$

which is (12). Using Proposition 2.4 we finish the proof of (ii). □

4. Positively homogeneous approximations

First, we present a quantitative and slightly more general version of [5, Theorem 3.4].

Theorem 4.1. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces, let $\bar{x} \in X$, and let α, μ, and δ be positive constants. Consider a continuous mapping $f : X \to Y$, with a closed convex domain $K \ni \bar{x}$, and a mapping $H : X \times X \rightrightarrows Y$ such that for each $x \in B_X(\bar{x}, \delta) \cap K$ the following conditions hold true:*

(i) *The mapping $H(x, \cdot)$ is positively homogeneous;*
(ii) *For each $y^* \in S_{Y^*}$ there is a non-zero $h \in B_X \cap \operatorname{cone}(K - x)$ such that $\inf \langle y^*, H(x, h)\rangle \geq \alpha + \mu$;*
(iii) *For each $h \in B_X$ there is a $\tau > 0$ such that for each $t \in (0, \tau)$, with $x + th \in B_X(\bar{x}, \delta) \cap K$, we have that $f(x + th) - f(x) \in H(x, th) + \mu t B_Y$.*

Assume finally that one of the following three conditions holds:

(a) *For each $x \in B_X(\bar{x}, \delta) \cap K$ and each $h \in S_X$ the set $H(x, h)$ is bounded and Y is either reflexive or the norm on Y is Fréchet smooth;*
(b) *For each $x \in B_X(\bar{x}, \delta) \cap K$ and each $h \in S_X$ the set $H(x, h)$ is relatively norm-compact and Y is either separable or the norm on Y is Gateaux smooth;*
(c) *For each $x \in B_X(\bar{x}, \delta) \cap K$ and each $h \in S_X$ the set $H(x, h)$ is relatively weakly compact and the norm on Y is weakly Hadamard smooth.*

Then $f\big(B_X(\tilde{x}, t) \cap K\big) \supset B_Y(f(\tilde{x}), ct)$ for each $\tilde{x} \in B_X(\bar{x}, \delta) \cap K$, each $c \in (0, \alpha)$, and each $t \in (0, \delta - \|\tilde{x} - \bar{x}\|]$.

Now, we present a continuous version of Theorem 4.1.

Theorem 4.2. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces, let $\bar{x} \in X$, let α, λ, μ, and δ be positive constants. Consider a mapping $f : X \to Y$, which has a closed convex domain $K \ni \bar{x}$ and is uniformly continuous on $B_X[\bar{x}, \delta] \cap K$, along with a mapping $H : X \times X \rightrightarrows Y$ such that for each $x \in B_X(\bar{x}, \delta) \cap K$ the following conditions hold true:*

(i*) *The mapping $H(x, \cdot)$ is positively homogeneous and*

$$H(x, u + v) \subset \overline{H(x, u) + H(x, v)} \quad \text{for each} \quad u, v \in X; \tag{19}$$

(ii) *For each $y^* \in S_{Y^*}$ there is a non-zero $h \in B_X$ such that $x + \lambda h \in K$ and $\inf \langle y^*, H(x, h)\rangle \geq \alpha + \mu$;*

(iii) *For each $h \in B_X$ and each $t \in (0, \lambda)$ such that $x + th \in B_X(\bar{x}, \delta) \cap K$, we have that $f(x + th) - f(x) \in H(x, th) + \mu t B_Y$;*

(iv) *For each $\varepsilon > 0$ there is a $\beta > 0$ such that $H(u, h) \subset H(x, h) + \varepsilon B_Y$ whenever $h \in B_X$ and $u \in B_X(x, \beta) \cap K$ with $u + \lambda h \in K$.*

Assume finally that one of the following two conditions holds:

(a*) *The set $H(B_X(\bar{x}, \delta) \cap K, S_X)$ is bounded and either Y is superreflexive or the norm on Y is uniformly Fréchet smooth;*

(b*) *The set $H(B_X(\bar{x}, \delta) \cap K, S_X)$ is relatively norm-compact and either Y is separable or the norm on Y is uniformly Gateaux smooth.*

Then for each $c \in (0, \alpha)$, each $\tilde{x} \in B_X(\bar{x}, \delta) \cap K$, and each $r \in (0, \delta - \|\tilde{x} - \bar{x}\|)$, there is $\widetilde{\varphi} \in \mathcal{C}(B_Y[f(\tilde{x}), cr]; K)$ such that $f(\widetilde{\varphi}(y)) = y$ and $c\|\widetilde{\varphi}(y) - \tilde{x}\| \leq \|y - f(\tilde{x})\|$ for each $y \in B_Y[f(\tilde{x}), cr]$.

Remark 4.3.

1. If we replace $B_X(\bar{x}, \delta)$ by $B_X[\bar{x}, \delta]$ in *all the assumptions* of Theorem 4.2, then the conclusion holds also for $r := \delta - \|\bar{x} - \tilde{x}\|$.
2. Often, we can assume that $\mu := 0$ in Theorem 4.1 and Theorem 4.2. Indeed, suppose that there is a constant $\mu > 0$ such that $f(x + h) - f(x) \in H(x, h) + \mu\|h\|B_Y$ for each $(x, h) \in X \times X$. Then $\widetilde{H}(x, h) := H(x, h) + \mu\|h\|B_Y$, $x, h \in X$, satisfies (iii) with $\mu := 0$. If $H(x, \cdot)$ satisfies (i), (i*), (ii), or (iv) then so does $\widetilde{H}(x, \cdot)$ with $\mu := 0$. If $H(x, h)$ is bounded [relatively norm-compact] then $\widetilde{H}(x, h)$ is bounded [has the measure of non-compactness less or equal to μ].

5. Approximations generated by sets of linear operators

The key tool is a slightly extended version of [5, Proposition 3.7].

Proposition 5.1. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces, let $K \subset X$ be a closed convex set with $0 \in K$, let $\alpha > 0$, and let $y^* \in S_{Y^*}$. Consider a convex set $\mathcal{A} \subset \mathcal{L}(X, Y)$ such that*

$$A(B_X \cap K) \supset \alpha B_Y \quad \text{for every} \quad A \in \mathcal{A}, \tag{20}$$

or only that

$$\sup \langle A^* y^*, B_X \cap K \rangle \geq \alpha \quad \text{for every} \quad A \in \mathcal{A}. \tag{21}$$

If X is reflexive, then there exists a non-zero $h \in B_X \cap K$ such that $\inf \langle y^*, \mathcal{A}h \rangle \geq \alpha$. *If the set $\mathcal{A}^* y^*$ is weak* compact and $\varepsilon \in (0, \alpha)$, then there exists a non-zero $h \in B_X \cap K$ such that* $\inf \langle y^*, \mathcal{A}h \rangle \geq \alpha - \varepsilon$.

Remark 5.2.
1. Assume that $\mathcal{A} \subset \mathcal{L}(X, Y)$ is compact in WOT, then the set $\mathcal{A}^* y^*$ is weak* compact for every $y^* \in Y^*$. This follows immediately from the WOT-to-weak* continuity of the assignment $\mathcal{A} \ni A \longmapsto A^* y^*$;
2. In reflexive spaces, the statement above can be found in [12, Lemma 2], and was inspired by Clarke's proof of [6, Lemma 3]. Proposition 5.1 can be generalized for bounded fans using the notion of the adjoint fan [13];

A bounded linear operator between Banach spaces is surjective if and only if $\sigma(A) > 0$, where

$$\sigma(A) := \sup\{c > 0 : \ A(B_X) \supset cB_Y\}.$$

The function $(\mathcal{L}(X, Y), \|\cdot\|) \ni A \longmapsto \sigma(A)$ is continuous (even Lipschitz with the constant 1), see [12, Corollary 6], [15, Lemma 1] or [17, Lemma 2.1]. This also follows from the Graves' theorem [9, Theorem 5D.2], which was generalized to mappings with closed convex domains, cf. [5, Theorem 3.9]. Hence, in particular, given a closed convex subset $K \ni 0$ of X, the function

$$(\mathcal{L}(X, Y), \|\cdot\|) \ni A \longmapsto \sigma_K(A) := \sup\{c > 0 : \ A(B_X \cap K) \supset cB_Y\}$$

is continuous (even locally Lipschitz).

Let us present a generalization of [2, Theorem 4.1], for the case that the solutions are requested to lie in a prescribed closed convex set.

Theorem 5.3. *Let $m \leq n$ be positive integers and let $\bar{x} \in \mathbb{R}^n$ be given. Consider a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$, Lipschitz continuous around $\bar{x}$, and a closed convex subset K of $\mathbb{R}^n$ containing $\bar{x}$. Assume that $\sigma_{K-\bar{x}}(A) > 0$ for each A from the Clarke generalized Jacobian $\partial f(\bar{x}) \subset \mathbb{R}^{m \times n}$. Then*

$$\bar{\sigma} := \min\{\sigma_{K-\bar{x}}(A) : \ A \in \partial f(\bar{x})\} > 0,$$

and for each $c \in (0, \bar{\sigma})$ there exist a neighborhood V of $f(\bar{x})$ and a `continuous` *mapping $\widetilde{\varphi} : V \to K$ such that $f(\widetilde{\varphi}(y)) = y$ and $c \|\widetilde{\varphi}(y) - \bar{x}\| \leq \|y - f(\bar{x})\|$ for each $y \in V$.*

Proof. The set $\partial f(\bar{x})$ is compact by [7, Theorem 2.6.2 (a)] and the function $\sigma_{K-\bar{x}}$ on $\partial f(\bar{x})$ is (Lipschitz) continuous. Hence the minimum in the definition of $\bar{\sigma}$ exists and is positive. Let $c \in (0, \bar{\sigma})$ be arbitrary. Pick an $\varepsilon > 0$ such that $c < \bar{\sigma} - 2\varepsilon$. Thus $\varepsilon < \bar{\sigma}/2$. As $\partial f(\bar{x})$ is a bounded set, there is an $s > 0$ such that

$$\mathcal{A} := \partial f(\bar{x}) + \varepsilon B_{\mathbb{R}^{n\times m}} \subset s B_{\mathbb{R}^{n\times m}}.$$

Find $\delta \in (0, \varepsilon/(s + \bar{\sigma}))$ such that f is Lipschitz continous on $B_{\mathbb{R}^n}[\bar{x}, \delta]$ and that $\partial f(z) \subset \mathcal{A}$ for each $z \in B_{\mathbb{R}^n}(\bar{x}, \delta)$; the latter is possible thanks to the upper semi-continuity of ∂f [7, Proposition 2.6.2 (c)]. Define $H(x, h) := \mathcal{A}h$, $(x, h) \in B_{\mathbb{R}^n}[\bar{x}, \delta] \times \mathbb{R}^n$. Note that $\mathcal{A}$ is a convex set. For all u, $x \in B_{\mathbb{R}^n}(\bar{x}, \delta) \cap K$, the Lipschitz mean value theorem [7, Proposition 2.6.5] implies that

$$f(u) - f(x) \in \operatorname{co}\big(\partial f([u, x])\big)(u - x) \subset \mathcal{A}(u - x) = H(x, u - x).$$

(Here we stress that, given a set $M \subset \mathbb{R}^n$, the symbol $\partial f(M)$ always means the union $\bigcup_{x\in M} \partial f(x)$.) Therefore (i*), (iii) with $\mu := 0$ and $\lambda := 1$, (iv), and (a*) in Theorem 4.2 are satisfied. We will show that (ii) therein holds as well. So, fix any $x \in B_{\mathbb{R}^n}(\bar{x}, \delta) \cap K$ and any $y \in S_{\mathbb{R}^m}$. Since $0 \in K - \bar{x}$ and $\sigma_{K-\bar{x}}(A) > \bar{\sigma} - \varepsilon$ for each $A \in \mathcal{A}$, Proposition 5.1, with $\alpha := \bar{\sigma} - \varepsilon$, implies that there is a $u \in B_{\mathbb{R}^n} \cap (K - \bar{x})$ such that $\inf \langle y, \mathcal{A}u\rangle \geq \bar{\sigma} - \varepsilon$. Let $h := (1 - \delta)u + (\bar{x} - x)$. As $\delta < \varepsilon/\bar{\sigma} < 1/2$, we have that $\|h\| < (1 - \delta) + \delta = 1$; that

$$x + h = \bar{x} + (1 - \delta)u \in \bar{x} + (1 - \delta)(K - \bar{x}) \subset \bar{x} + K - \bar{x} = K;$$

and also that

$$\begin{aligned}\inf \langle y, \mathcal{A}h\rangle &\geq (1 - \delta) \inf \langle y, \mathcal{A}u\rangle - s\|x - \bar{x}\| > (1 - \delta)(\bar{\sigma} - \varepsilon) - s\delta \\ &> \bar{\sigma} - \varepsilon - \delta(\bar{\sigma} + s) > \bar{\sigma} - 2\varepsilon.\end{aligned}$$

As $H(x, h) = \mathcal{A}h$ for each $x \in B_{\mathbb{R}^n}(\bar{x}, \delta)$, we showed (ii) with $\lambda := 1$, $\alpha := \bar{\sigma} - 2\varepsilon$, and $\mu := 0$. Now Theorem 4.2, with $r := \delta/2$ and $\tilde{x} := \bar{x}$, finishes the proof. □

Finally, we derive [2, Theorem 5.1], which is a semi-local version of the above statement, that is, the size of the neighborhoods is prescribed. For simplicity we consider $K = \mathbb{R}^n$ only, but a generalization to arbitrary closed convex domain is immediate.

Theorem 5.4. *Let $m \leq n$ be positive integers, let δ be a positive constant, and let $\bar{x} \in \mathbb{R}^n$ be given. Consider a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$, which is locally Lipschitz continuous on an open set $U \supset B_{\mathbb{R}^n}[\bar{x}, \delta]$ and such that*

$$\bar{\sigma} := \min \big\{\sigma(A) : \ A \in \partial f(x),\ x \in B_{\mathbb{R}^n}[\bar{x}, \delta]\big\} > 0.$$

Then for each $c \in (0, \bar{\sigma})$ and each $\tilde{x} \in B_{\mathbb{R}^n}(\bar{x}, \delta)$ there is a `continuous` *mapping $\widetilde{\varphi} : B_{\mathbb{R}^m}[f(\tilde{x}), cr] \to \mathbb{R}^n$ such that $f(\widetilde{\varphi}(y)) = y$ and $c\,\|\widetilde{\varphi}(y) - \tilde{x}\| \leq \|y - f(\tilde{x})\|$ for each $y \in B_{\mathbb{R}^m}[f(\tilde{x}), cr]$, where $r := \delta - \|\tilde{x} - \bar{x}\|$.*

Proof. Fix any $c \in (0,\bar\sigma)$. Put $\Omega := B_{\mathbb{R}^n}[\bar x,\delta]$. As f is locally Lipschitz continuous on U, it is Lipschitz continuous on the (compact) set Ω. Hence there is an $s>0$ such that $\partial f(\Omega) \subset sB_{\mathbb{R}^{m\times n}}$. By [7, Proposition 2.6.2], ∂f is upper semi-continuous with compact (convex) values. This justifies taking "min" in the definition of $\bar\sigma$.

Consider any $\lambda \in (0,\infty)$. Find a finite set $F_\lambda \subset \Omega$ such that $\bigcup_{y\in F_\lambda} B_{\mathbb{R}^n}(y,\lambda)$ contains Ω. For $x,y\in\Omega$ put

$$\gamma_{\lambda,y}(x) := \operatorname{dist}\big(x, \Omega\setminus B_{\mathbb{R}^n}(y,\lambda)\big)$$

and then

$$\nu_{\lambda,y}(x) := \gamma_{\lambda,y}(x)\Big/\textstyle\sum_{z\in F_\lambda}\gamma_{\lambda,z}(x).$$

It is easy to check that $\nu_{\lambda,y}(\cdot)$'s are continuous (even locally Lipschitz) functions. Also $\sum_{y\in F_\lambda}\nu_{\lambda,y}(x)=1$ for every $x\in\Omega$. Finally, define the mapping $\mathcal{A}_\lambda : \Omega \rightrightarrows \mathbb{R}^{m\times n}$ by

$$\mathcal{A}_\lambda(x) := \sum_{y\in F_\lambda}\nu_{\lambda,y}(x)\,\overline{\operatorname{co}}\,\partial\big(B_{\mathbb{R}^n}(y,2\lambda)\cap\Omega\big), \quad x\in\Omega.$$

Clearly, $\mathcal{A}_\lambda$ has compact convex values. It is left for a reader to verify that $\mathcal{A}_\lambda : \Omega \rightrightarrows \mathbb{R}^{m\times n}$ is upper (and also lower) semi-continuous. Further, we note that, given any $x\in\Omega$, we have

$$\partial f\big(B_{\mathbb{R}^n}(x,\lambda)\cap\Omega\big) \subset \mathcal{A}_\lambda(x) \subset \overline{\operatorname{co}}\,\partial f\big(B_{\mathbb{R}^n}(x,3\lambda)\cap\Omega\big) \quad (\subset sB_{\mathbb{R}^{m\times n}}). \tag{22}$$

Indeed, for each $y\in F_\lambda$ such that $\nu_{\lambda,y}(x)>0$ we have $x\in B_{\mathbb{R}^n}(y,\lambda)$; thus $B_{\mathbb{R}^n}(x,\lambda)\subset B_{\mathbb{R}^n}(y,2\lambda)\subset B_{\mathbb{R}^n}(x,3\lambda)$, which implies both the inclusions.

Pick an $\alpha\in(c,\bar\sigma)$; say $\alpha := (c+\bar\sigma)/2$. We `claim` that there is a $\lambda\in(0,\infty)$ such that for each $u\in\Omega$ and each $A\in\mathcal{A}_\lambda(u)$ we have $\sigma(A)\ge\alpha$. Suppose, on the contrary, that for each $i\in\mathbb{N}$, there are $u_i\in\Omega$ and $A_i\in\mathcal{A}_{1/i}(u_i)$ such that $\sigma(A_i)<\alpha$. Assume, without any loss of generality, that the sequence (u_i) converges to a $u\in\Omega$. From the upper semi-continuity of ∂f at u, find an $i\in\mathbb{N}$ so big that

$$\partial f\big(B_{\mathbb{R}^n}(u,3/i+\|u-u_i\|)\big) \subset \partial f(u) + (\bar\sigma-\alpha)B_{\mathbb{R}^{m\times n}}.$$

Then, using the latter inclusion in (22), we get

$$\begin{aligned} A_i &\in \mathcal{A}_{1/i}(u_i) \subset \overline{\operatorname{co}}\,\partial f\big(B_{\mathbb{R}^n}(u_i,3/i)\cap\Omega\big) \\ &\subset \overline{\operatorname{co}}\,\partial f\big(B_{\mathbb{R}^n}(u,3/i+\|u-u_i\|)\big) \subset \partial f(u) + (\bar\sigma-\alpha)B_{\mathbb{R}^{m\times n}}. \end{aligned}$$

Thus, there is an $A\in\partial f(u)$ such that $\|A-A_i\|\le\bar\sigma-\alpha$. Therefore, using a well known fact that the function $\sigma(\cdot)$ is 1-Lipschitzian, we get that

$$\alpha > \sigma(A_i) \ge \sigma(A) - \|A-A_i\| \ge \bar\sigma - (\bar\sigma-\alpha) = \alpha,$$

a contradiction.

Now we are ready to apply Theorem 4.2, by verifying all the assumptions therein. Consider the α defined in the previous paragraph and keep the λ found in the Claim above. Put $\mu := 0$, $K := \mathbb{R}^n$, and $H(x,h) := \mathcal{A}_\lambda(x)h$ for each $(x,h) \in \Omega \times \mathbb{R}^n$. Then the assumptions (i*), (iv), and (a*) in Theorem 4.2 are clearly satisfied. To verify (ii), fix any $x \in \Omega$ and any $y \in S_{\mathbb{R}^m}$. As $\alpha < \bar{\sigma}$, Proposition 5.1, with $\mathcal{A} := \mathcal{A}_\lambda(x)$, yields an $h \in B_{\mathbb{R}^n}$ such that $\inf \langle y, H(x,h)\rangle = \inf \langle y, \mathcal{A}_\lambda(x)h\rangle \geq \alpha$. Trivially $x + \lambda h \in K = \mathbb{R}^n$. Concerning (iii), pick arbitrary $t \in (0,\lambda)$ and $h \in B_{\mathbb{R}^n}$ with $x + th \in \Omega$. The Lipschitz mean value theorem [7, Proposition 2.6.5] and the first inclusion in (22) imply that

$$\begin{aligned} f(x+th) - f(x) &\in \big(\operatorname{co} \partial f\,[x, x+th]\big)(th) \\ &\subset \Big(\overline{\operatorname{co}}\, \partial f\big(B_{\mathbb{R}^n}(x,\lambda) \cap \Omega\big)\Big)\,(th) \subset \mathcal{A}_\lambda(x)(th) = H(x,th). \end{aligned}$$

Apply Theorem 4.2 to finish the proof (see Remark 4.3.1). □

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NTIS - New Technologies for the Information Society and Department of Mathematics, Faculty of Applied Sciences, University of West Bohemia, Univerzitní 8, 301 00 Plzeň, Czech Republic

E-mail address: cibi@kma.zcu.cz

Žitná 25, 115 67 Praha 1, Czech Republic

E-mail address: fabian@math.cas.cz