An algorithm for pseudo-monotone operators with application to rational approximation

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Notations

1. $\mathbb{R}^n$: finite dimensional Euclidean space
2. $\langle \cdot, \cdot \rangle$: the inner product
3. $|| \cdot ||$: the induced norm
4. $| \cdot |$: the absolute value
5. $convD$ the convex hull of the set D
6. $linD$ the linear span of the set D
7. $Gr(T)$ the graph of T: $\{(x, u) \in \mathbb{R}^n \times \mathbb{R}^n \quad : u \in T(x)\}$.
8. $\mathbb{P}_n$: The set of polynomials with degree $\leq n$
9. $\mathbb{F}_n$: $lin\{h_0(t), h_1(t), h_2(t), h_3(t), \cdots, h_n(t)\}$, $h_i : \mathbb{R} \to \mathbb{R}$ is a function.

The VIP(T,C)

Consider the operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and the set $C \subseteq \mathbb{R}^n$, the variational inequality problem for T and C, denoted by VIP(T,C), is defined as:

$$\text{Find } x^* \in C : \exists u^* \in T(x^*) : \langle u^*, x - x^* \rangle \geq 0, \forall x \in C, \qquad (1)$$

we denote the solution set of problem (1), by S_*.

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Related with the VIP(T,C) is the dual variational inequality problem (DVIP(T,C)):

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which solution set is denoted by S_0.
When the operator T is pseudo-monotone, it is known that both problems are equivalents, i.e., $S_* = S_0$. But there are examples (see Burachik, R. and Díaz Millán, R.) for which $S_* \neq S_0$.

Preliminaries results

Definition

Given a convex, closed and non-empty subset $C \subseteq \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$. We define the projection of x onto C, denoted by $P_C(x)$, by the unique solution of the problem

$$\min_{z \in C} \|z - x\|.$$

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Orthogonal Projection Properties

Let $C \subseteq \mathbb{R}^n$ be closed and convex. For all $x, y \in \mathbb{R}^n$ and all $z \in C$, the following holds:

1. $\|P_C(x) - P_C(y)\|^2 \leq \|x - y\|^2 - \|(x - P_C(x)) - (y - P_C(y))\|^2$.
2. $\langle x - P_C(x), z - P_C(x) \rangle \leq 0$.

Preliminaries results

Definition

Let S be a nonempty subset of $\mathbb{R}^n$. A sequence $(x^k)_{k\in\mathbb{N}} \subset \mathbb{R}^n$ is said to be Fejér convergent to S if and only if for all $x \in S$ there exists $k_0 \in \mathbb{N}$ such that $||x^{k+1} - x|| \leq ||x^k - x||$ for all $k \geq k_0$.

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Fact

If $(x^k)_{k\in\mathbb{N}}$ is Fejér convergent to S, then the following hold

1. *The sequence $(x^k)_{k\in\mathbb{N}}$ is bounded.*
2. *The sequence $(\|x^k - x\|)_{k\in\mathbb{N}}$ converges for all $x \in S$.*
3. *If an accumulation point x_* belongs to S, then the sequence $(x^k)_{k\in\mathbb{N}}$ converges to x_*.*

Preliminaries results

Definition

A point-to-set operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called:

- *Monotone*, iff for all $(x, u), (y, v) \in Gr(T)$,

$$\langle u - v, x - y \rangle \geq 0.$$

- *Pseudo-monotone*, iff for all $(x, u), (y, v) \in Gr(T)$, the following implication holds:

$$\langle u, y - x \rangle \geq 0 \Longrightarrow \langle v, y - x \rangle \geq 0.$$

- *Quasi-monotone*, iff for all $(x, u), (y, v) \in Gr(T)$, the following implication holds:

$$\langle u, y - x \rangle > 0 \Longrightarrow \langle v, y - x \rangle \geq 0.$$

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It is clear that every monotone operator is pseudo-monotone, and every pseudo-monotone operator is quasi-monotone.

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The Linesearch

Now, we present a modification of the Linesearch F, used in Burachik-Millán, suitable for our problem:

Linesearch

Input: $x \in C$, $\beta > 0$ and $\delta \in (0,1)$. Set $\alpha \leftarrow 1$ and $\theta \in (0,1)$. Define $z = P_C(x - \beta u)$ with $u \in T(x)$. **If** for all $u \in T(x)$

$$\min_{u^\alpha \in T(x_\alpha)} \langle u^\alpha, x - z \rangle < \delta \langle u, x - z \rangle,$$

where $x_\alpha = \alpha z + (1-\alpha)x$,
then $\alpha \leftarrow \theta\alpha$, **Else** Stop and choose $u^\alpha \in T(x_\alpha)$:
$\langle u^\alpha, x - z \rangle \geq \delta \langle u, x - z \rangle$
Output: (α, u^α).

Agorithm F

Given $(\beta_k)_{k\in\mathbb{N}} \subset [\check{\beta}, \hat{\beta}]$ such that $0 < \check{\beta} \le \hat{\beta} < +\infty$ and $\delta \in (0,1)$.
Initialization: Take $x^0 \in C$ and set $k \leftarrow 0$. **Step 1:** Set $z^k = P_C\left(x^k - \beta_k u^k\right)$ with $u^k \in T(x^k)$ and

$$(\alpha_k, u^{\alpha_k}) = \textbf{Linesearch}(x^k, \beta_k, \delta),$$

Step 2 (Stopping Criterion): If $z^k = x^k$ or $x^k = P_C\left(x^k - v^k\right)$ with $v^k \in T(x^k)$, then stop. Otherwise,
Step 3: Set

$$\bar{x}^k := \alpha_k z^k + (1-\alpha_k)x^k, \tag{3a}$$

$$\text{and} \quad x^{k+1} := \mathcal{F}(x^k); \tag{3b}$$

Step 4: If $x^{k+1} = x^k$, stop. Otherwise, set $k \leftarrow k+1$ and go to **Step 1**.

The algorithm

We consider two variants. Their main difference lies in the computation (3b):

$$\mathcal{F}_1(x^k) = P_C\left(P_{H(\bar{x}^k, u^{\alpha_k})}(x^k)\right); \quad (\textbf{Variant 1}) \tag{4}$$

$$\mathcal{F}_2(x^k) = P_{C \cap H(\bar{x}^k, u^{\alpha_k})}(x^k); \quad (\textbf{Variant 2}) \tag{5}$$

where $u^{\alpha_k} \in T(\bar{x}^k)$ and

$$H(x, u) := \{y \in \mathbb{R}^n : \langle u, y - x \rangle \le 0\} \tag{6}$$

Assumptions

Assumptions

We assume that the operator $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfy the following conditions:

A1) T is closed.

A2) T is bounded on bounded sets.

A3) The solution sets of the dual and primal problems coincide ($S_0 = S_*$).

A1) The operator is *closed* when the graph is closed.

A2) Classical assumption in the literature.

A3) Weaker that pseudo-monotone. If T is pseudo-monotone and A1, then A3 is satisfied.

Convergence

Proposition

If $x \in C$ is not a solution of Problem (1), ***Linesearch*** *terminates after finitely many iterations.*

Proposition

$x^k \in S_* \leftrightarrow x^k \in H(\bar{x}^k, u^{\alpha_k})$.

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Proposition

If the algorithm stops in a finite number of iterations, then stops in a solutions set.

Convergence

Proposition for both Variants

$(x^k)_{n\in\mathbb{N}}$ generated by **Algorithm F**. The following hold:

1. The sequence $(x^k)_{k\in\mathbb{N}}$ is Fejér convergent to S_*.
2. The sequence $(x^k)_{k\in\mathbb{N}}$ is bounded.
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Variant 2

1. $x^{k+1} = P_{C\cap H(\bar{x}^k,u^{\alpha_k})}(x^k) = P_{C\cap H(\bar{x}^k,u^{\alpha_k})}\left(P_{H(\bar{x}^k,u^{\alpha_k})}(x^k)\right)$.
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2. $\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0$.

Theorem for both Variants

The sequences generated by both variants of the **Algorithm F** converges to a point in S_*.

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Approximation of continuous functions

Consider the continuous function $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is a compact set.
We are interested in approximate f by a function $\frac{p}{q}$ where $p \in \mathbb{F}_n$ and $q \in \mathbb{F}_m$, over the set I. In others words, we want to solve the optimisation problem:

$$\min_{p \in \mathbb{F}_n, q \in \mathbb{F}_m} \sup_{t \in I} \left| f(t) - \frac{p(t)}{q(t)} \right|. \tag{7}$$

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We denote $p(t) = \langle \mathbf{a}, \mathbf{p}_{\mathbf{t}_n} \rangle = a_0 p_0(t) + a_1 p_1(t) + a_2 p_2(t) \cdots + a_n h_n(t)$,
where $\mathbf{a} = (a_0, a_1, a_2, \ldots, a_n) \in \mathbb{R}^{n+1}$, and
$\mathbf{p}_{\mathbf{t}_n} = (p_0(t), p_1(t), p_2(t), \ldots, p_n(t)) \in \mathbb{R}^{n+1}$ for each $t \in I$.
$q(t) = \langle \mathbf{b}, \mathbf{q}_{\mathbf{t}_m} \rangle = b_0 q_0(t) + b_1 q_1(t) + \cdots + b_m q_m(t)$.
where $p_i, q_j : \mathbb{R} \to \mathbb{R}$ are real functions, for all $i, j \in \mathbb{N}$.

Problem (7) can be written as:

$$\min_{(\mathbf{a},\mathbf{b})\in C} \Psi^f(\mathbf{a},\mathbf{b}), \tag{8}$$

where $\Psi^f(\mathbf{a},\mathbf{b}) = \sup_{t\in I}\left|f(t) - \frac{\langle \mathbf{a},\mathbf{p}_{\mathbf{t}_n}\rangle}{\langle \mathbf{b},\mathbf{q}_{\mathbf{t}_m}\rangle}\right|$ is a maximal deviation of f in I, and $C = \{(\mathbf{a},\mathbf{b}) \in \mathbb{R}^{n+1} \times \mathbb{R}^{m+1} : \langle \mathbf{b}, \mathbf{t}_m\rangle \geq 1, \forall t \in I\}$.

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If for all $i, j \in \mathbb{N}$ we define $p_i(t) = q_i(t) = t^i$, the problems (7) and (8) becomes in the classical *Rational Approximation* of a continuous function.

Definitions and Results

We denote by $A^+(\mathbf{a},\mathbf{b})$ and $A^-(\mathbf{a},\mathbf{b})$ the sets of actives values, i.e.,
$A^+(\mathbf{a},\mathbf{b}) = \left\{t \in I : \Psi^f(\mathbf{a},\mathbf{b}) = \sigma_t^f(\mathbf{a},\mathbf{b})\right\}$ and
$A^-(\mathbf{a},\mathbf{b}) = \left\{t \in I : \Psi^f(\mathbf{a},\mathbf{b}) = -\sigma_t^f(\mathbf{a},\mathbf{b})\right\}$, where
$\sigma_t^f(\mathbf{a},\mathbf{b}) := f(t) - \frac{\langle \mathbf{a},\mathbf{p}_{\mathbf{t}_n}\rangle}{\langle \mathbf{b},\mathbf{q}_{\mathbf{t}_m}\rangle}$.

Lemma

For all real function $f : \mathbb{R} \to \mathbb{R}$ and $I \subseteq \mathbb{R}$ compact, the maximal deviation $\Psi^f : \mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \to \mathbb{R}$ is a quasi-convex function on C.

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Proof

For all $t_0 \in I$, the function $\left| f(t_0) - \frac{\langle \mathbf{a},\mathbf{p}^{\mathbf{0}}_{\mathbf{t}\,n}\rangle}{\langle \mathbf{b},\mathbf{q}^{\mathbf{0}}_{\mathbf{t}\,m}\rangle} \right|$, is a quasi-linear function, then it is a quasi-convex. Since Ψ^f is the supremum of quasi-convex functions, it is quasi-convex.

Definitions and Results

Theorem (Rockafellar and Wets book)

The Clarke subdifferential of the function Ψ^f can be computed as follow:

$$\partial\Psi^f(\mathbf{a},\mathbf{b}) = conv\left\{\nabla\sigma_t^f(\mathbf{a},\mathbf{b}), -\nabla\sigma_l^f(\mathbf{a},\mathbf{b}) : t \in A^+(\mathbf{a},\mathbf{b}), l \in A^-(\mathbf{a},\mathbf{b})\right\}. \quad (9)$$

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Proposition

Given a continuous function f, if $0 \in \partial\Psi^f(\mathbf{a},\mathbf{b})$ then $(\mathbf{a},\mathbf{b})$ is a global minimizer of Ψ^f.

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Theorem

The function Ψ^f is a pseudo-convex function. Consequently the Clarke Subdifferential $\partial \Psi^f$ is a pseudomonotone operator.

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Some facts

Since $\partial\Psi^f$ is a pseudo-monotone operator, taking $T = \partial\Psi^f$ we have $S_0 = S_*$, satisfying (A3).

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It is well know that problems (8) and (1) when $T = \partial\Psi^f$ are equivalents. Then, solving problem (8) is enough to solve problem (1).

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It is well know that problems (8) and (1) when $T = \partial\Psi^f$ are equivalents. Then, solving problem (8) is enough to solve problem (1).
How can we implement the **Linesearch F**?

Linesearch F for Problem (8)

Input: $(a, b) \in C$, $\beta > 0$ and $\delta \in (0, 1)$.
Set $\alpha \leftarrow 1$ and $\theta \in (0, 1)$.
For all $(u_a, u_b) \in \left\{\nabla\sigma_t^f(a, b), -\nabla\sigma_l^f(a, b) : t \in A^+(a, b), l \in A^-(a, b)\right\}$,
define $(z_a, z_b) = P_C((a, b) - \beta(u_a, u_b))$.
If for each (u_a, u_b) we have

$$\min_{(u_a^\alpha, u_b^\alpha) \in D_\alpha} \langle (u_a^\alpha, u_b^\alpha), (a, b) - (z_a, z_b) \rangle < \delta \langle (u_a, u_b), (a, b) - (z_a, z_b) \rangle,$$

where
$D_\alpha := \left\{\nabla\sigma_t^f(a_\alpha, b_\alpha), -\nabla\sigma_l^f(a_\alpha, b_\alpha) : t \in A^+(a_\alpha, b_\alpha), l \in A^-(a_\alpha, b_\alpha)\right\}$
and
$(a_\alpha, b_\alpha) = \alpha(z_a, z_b) + (1 - \alpha)(a, b)$,
then $\alpha \leftarrow \theta\alpha$, **Else** Return α. **Output:** $\left(\alpha, (u_a^\alpha, u_b^\alpha)\right)$.

Rational Approximation with Polynomial

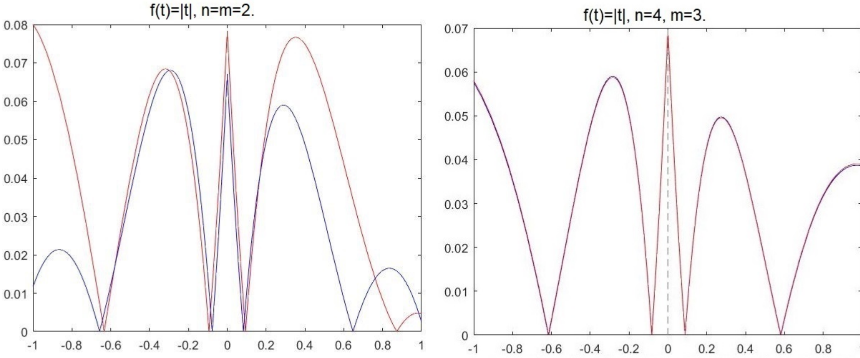
Unknown Solution on the interval

- 200 iterations and $I = [-1, 1]$.
- Ψ_i^f with $i = 1, 2$ denotes the function value for the Variants 1 and 2 respectively.
- *iter* is the iteration number in which was attained the best result.
- n, m represent the degree of the numerator and denominator polynomials respectively.
- In all pictures, the blue colour is for Variant 1, the red colour for Variant 2. In green colour is the graph of the function f.

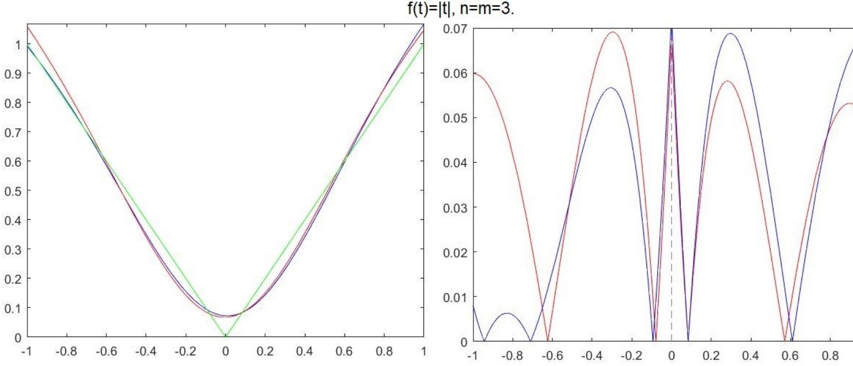
Results

Algorithm F for unknown solution					
$f(t)$	(n,m)	$Iter$	$\Psi_1^f(a,b)$	$Iter$	$\Psi_2^f(a,b)$
$\lvert t\rvert$	$(2,2)$	161	0.068	186	0.070
$\lvert t\rvert$	$(3,3)$	193	0.072	191	0.069
$\lvert t\rvert$	$(4,3)$	198	0.069	198	0.069
$\sin(t)$	$(2,2)$	186	0.0097	186	0.014
$\lvert\sin(t)\rvert$	$(3,3)$	189	0.0714	193	0.0700
$\sqrt{\lvert t\rvert}$	$(4,4)$	193	0.186	185	0.182

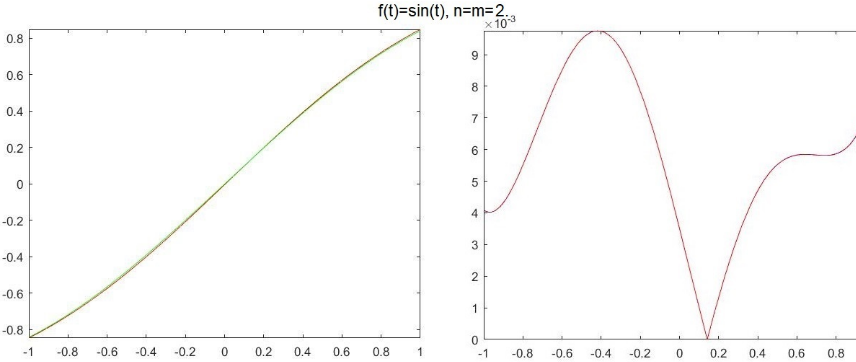
Objective function Ψ^f



f and p/q and Objective function Ψ^f

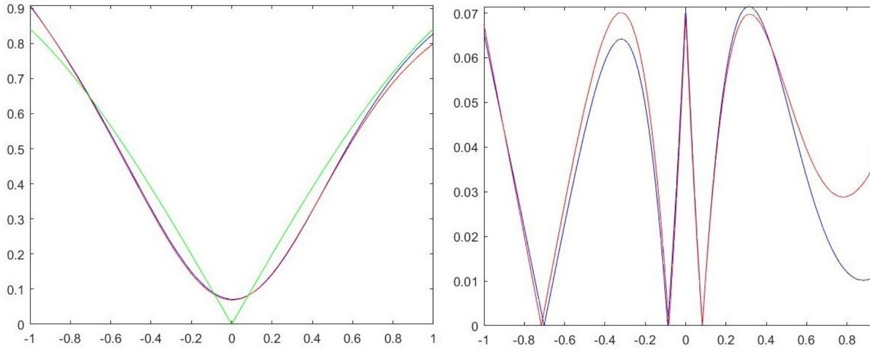


Objective function Ψ_i^f



f and p/q and Objective function Ψ^f

f(t)=|sin(t)|, n=m=3.



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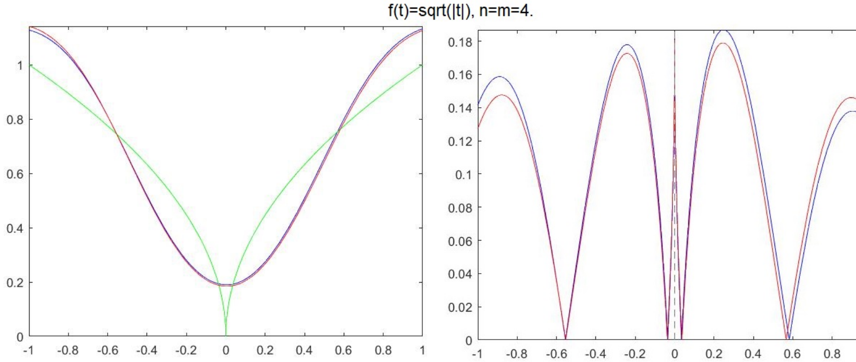

Known Solution over discrete I

In this subsection we testing rational functions as the objective function.

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Known Solution over discrete I

- $d_i(x, S^*)$ denotes the distance between the last point to the solution set for the variant i.
- We stopped the algorithm when the function value Ψ_i^f at the current point be less or equal to 10^{-3}.
- For the compact set I, we used a collection of M equidistant points on the interval $[-1, 1]$.

Results

Algorithm F for known solution						
$f(t)$	(n,m)	M	$Iter_1$	$d_1(x,S^*)$	$Iter_2$	$d_2(x,S^*)$
1	$(1,1)$	100	12	$9.7*10^{-4}$	12	$9.8*10^{-4}$
1	$(2,2)$	200	455	0.004	381	0.002
$\frac{1}{t^2+1}$	$(1,2)$	100	423	0.009	135	0.004
$\frac{1}{t^2+1}$	$(2,2)$	200	4406	0.097	3950	0.097
$\frac{t}{t+1.5}$	$(1,1)$	100	4240	0.013	3137	0.006
$\frac{t}{t+1.5}$	$(2,2)$	200	6490	0.017	5875	0.009
$\frac{t^2-1}{t+2}$	$(2,2)$	200	2361	0.02	1730	0.043
$\frac{t^2-1}{t+2}$	$(3,2)$	100	14643	0.09	6306	0.039

Non-polynomial rational approximation

In this section we consider different rational function to approximate continuous functions.

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Non-Polynomials Approximation

- *CPU* denotes the CPU time
- By $h(t)$, we denote the functions which compose the rational function
- For this subsection we consider as the objective function the continuous function $f(t) = \frac{\sin t - \cos t}{t+2}$.
- The compact set in all cases are the collection of M equidistant points in the interval $[-1, 1]$
- The stopping criteria is $\Psi^f(\mathbf{a}, \mathbf{b}) \leq 10^{-2}$.

Results

Algorithm F with non-polynomial						
$h(t)$	(n,m)	M	$Iter_1$	CPU_1	$Iter_2$	CPU_2
e^t	$(3,3)$	20	625	16.9	410	11.3
e^t	$(3,3)$	100	585	21.1	569	23.4
e^t	$(5,4)$	200	2115	420.5	2094	419.2
e^t	$(10,8)$	100	5121	997.8	5099	940.1
$\sin(t)$	$(3,3)$	20	255	12.0	255	11.4
$\sin(t)$	$(3,3)$	100	429	44.6	311	35.3
$\sin(t)$	$(5,4)$	200	235	52.1	184	37.4
$\sin(t)$	$(10,8)$	100	136	23.9	214	20.9

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Conclusions Remarks

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Open problems

- Can we extend **Algorithm F** to infinite dimensional spaces?
- Can we find an full implementable linesearch?
- Can we extend this algorithm for other Approximation Problems?

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Thanks!