On diametrically maximal sets, maximal premonotone mappings and premonotone bifunctions

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- If A is a bounded set, the diameter of A is denoted by $diam(A) = \sup\{\|x - y\| : (x, y) \in A \times A\}$.

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- If C is a diamax set with diameter $s > 0$, then the ball $B(c, r)$ is contained in C for any midpoint c of C and $r = (1 - \frac{\sqrt{3}}{2})s$. It implies that any diamax set, with diameter strictly positive, has nonempty interior.

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- Let $T \subset \mathbb{R}^n \times \mathbb{R}^n$ be a mapping. T is premonotone if and only if $\langle u - v, y - x \rangle \leq \min\{\sigma_T(x), \sigma_T(y)\} \|y - x\| < +\infty$ $\forall \{(x, u), (y, v)\} \subset T$

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- If T is premonotone, then there exists a premonotone mapping T', an extension of T, such that $T \subset T', \sigma_T(x) \leq \sigma_{T'}(x)$ $\forall x \in dom(T)$ and $int(dom(T'))$ is nonempty and convex.

Maximal premonotonicity

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- Note that $dom(G^h)$ is no convex and $dom(G^c)$ is convex.

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vi) If T, U are premonotone and $T \subset U$ then $\sigma_T(x) \leq \sigma_U(x)$ for all $x \in \mathbb{R}^n$.

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- If $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is premonotone, then $(T^c(x))^\infty = N_{D(T)}(x)$ for all $x \in cl(D(T))$, with $D(T) = int(co(dom((T)))$. And $(T^h(x))^\infty = (T^c(x))^\infty \ \forall x \in dom(T)$.

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- Let $U : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and strongly monotone with constant γ, $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ inner Lipschitz semicontinuous with constant β, and assume that $C(x)$ is compact for all $x \in \mathbb{R}^n$ and that $\gamma \geq \beta$. Define $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ as $T(x) = U(x) + C(x)$ for all $x \in \mathbb{R}^n$. Then $\sigma_T(y) = \mathrm{diam}(C(y))$ for all $y \in \mathbb{R}^n$.

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- Let $U : \mathbb{R}^n \to \mathbb{R}^n$ be a maximal monotone mapping and a compact set $\widehat{C} \subset \mathbb{R}^n$. The mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by $T(x) = U(x) + \widehat{C}$ is premonotone with $\sigma_T(x) = \text{diam}(\widehat{C})$ for all $x \in \mathbb{R}^n$. T is σ_T-maximal premonotone if and only if $\widehat{C}$ is a diamax set.

Remarks

- If $f : \Omega \to \mathbb{R}$ if a continuous differentiable function and $\epsilon > 0$, then the mapping $T : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(x) = \nabla f(x) + \epsilon S(x)$, where $S(x)_i = \sin(x_i)$ is premonotone in Ω

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- If $T \subset \mathbb{R}^n \times \mathbb{R}^n$ is premonotone, then T^c is the union of maximal monotone mappings.

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- Moreover, when the maximal mapping (in the conjecture) is integrable, then the premonotone mapping is generated by a perturbation of a convex function.

Premonotone bifunctions

- For each nonempty set $K \subset \mathbb{R}^n$, consider bifunctions $f : K \times \mathbb{R}^n \to \mathbb{R}$ satisfying
  B1. For each $x \in K$: $f(x,x) = 0$,
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- Let $f : K \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction satisfying assumptions $\mathrm{B1}$ and $\mathrm{B2}$. If f is σ-premonotone bifunction, then f satisfies $\mathrm{B3}$.

Premonotone bifunctions

- In order to build bifunctios from mappings, we need to consider the following properties for mappings $T \subset \mathbb{R}^n \times \mathbb{R}^n$, here $D(T) = int(co(dom(T))) \neq \emptyset$ and $D_T = D(T) \cap dom(T)$.

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- For each mapping $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with $D_T \neq \emptyset$ and satisfying A3, define the bifunction $f_T : D_T \times \mathbb{R}^n \to \mathbb{R}$ as:

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- Corollary
- Consider $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $f : K \times \mathbb{R}^n \to \mathbb{R}$.
  - i) If T is monotone, then f_T is monotone and satisfies $\mathrm{B1}$-$\mathrm{B3}$.
  - ii) If T is σ-premonotone, then f_T is σ-premonotone and satisfies $\mathrm{B1}$-$\mathrm{B3}$.
  - iii) If f is monotone and satisfies B2, then T_f is monotone and satisfies $\mathrm{A1}$-$\mathrm{A3}$.
  - iv) If f is σ-premonotone and satisfies B2, then T_f is σ-premonotone and satisfies $\mathrm{A1}$-$\mathrm{A3}$.

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- Proposition 15
- The restriction of the mapping F to Γ and the restriction of the mapping G to $F(\Gamma)$ are bijections and mutual inverses, meaning that $(F \circ G)(f) = f$ for all $f \in F(\Gamma)$ and $(G \circ F)(T) = T$ for all $T \in \Gamma$.

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