Necessary conditions for transversality properties

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joint work with **Alex Kruger** and **Hoa Bui**

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Variational Analysis and Optimisation Webinar

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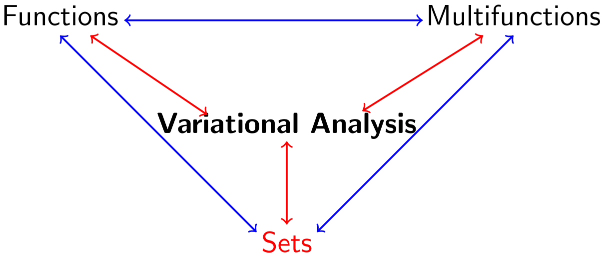
Outline

Introduction

Transversality, Regularity and Error Bounds in Variational Analysis and Optimisation



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Transversality: 'good' arrangements of collections of sets.

- nonsmooth calculus
- convergence analysis
- optimality conditions

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☐ Ioffe, A.D.: **Variational Analysis of Regular Mappings**. Theory and Applications. Springer Monographs in Mathematics. Springer (2017)

"**Regularity** is a property of **a single object** while **transversality** relates to the interaction of **two or more independent objects**."

Nonlinearity

- Let $\mathcal{C}$ be a family of all continuous strictly increasing functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\varphi(0) = 0$ and $\lim_{t\to+\infty} \varphi(t) = +\infty$.

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- Let $\widehat{\mathcal{C}}_\delta \subset \mathcal{C}$:

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  Example: $\varphi(t) := \alpha^{-1}t^q$, $q \in]0, 1]$

Semitransversality

X - normed space, $\bar{x} \in \Omega_1 \cap \Omega_2$, $\varphi \in \mathcal{C}$

Definition

$\{\Omega_1, \Omega_2\}$ is $\varphi-$semitransversal at $\bar{x}$ if

$$(\Omega_1 - x_1) \cap (\Omega_2 - x_2) \cap B_\rho(\bar{x}) \neq \emptyset$$

$\forall \rho \in]0, \delta[$, $x_1, x_2 \in X$ with $\varphi(\max_{i=1,2} \|x_i\|) < \rho$.

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$\forall \rho \in]0, \delta_1[$, $\omega_i \in \Omega_i \cap B_{\delta_2}(\bar{x})$, $x_i \in X$ $(i = 1, 2)$ with $\varphi(\max_{i=1,2} \|x_i\|) < \rho$.

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Metric Characterizations

- $\varphi-$semitransversality:

$$d\left(\bar{x}, (\Omega_1 - x_1) \cap (\Omega_2 - x_2)\right) \leq \varphi\left(\max_{i=1,2} \|x_i\|\right)$$

$\forall x_1, x_2 \in X$ with $\varphi(\max_{i=1,2} \|x_i\|) < \delta$.

- $\varphi-$subtransversality:

$$d\left(x, \Omega_1 \cap \Omega_2\right) \leq \varphi\left(\max_{i=1,2} d(x, \Omega_i)\right)$$

$\forall x \in B_{\delta_2}(\bar{x})$ with $\varphi\left(\max_{i=1,2} d(x, \Omega_i)\right) < \delta_1$.

- $\varphi-$transversality:

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Convex Case

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- $\{\Omega_1, \Omega_2\}$ is $\varphi-$semitransversal at $\bar{x}$ with δ if and only if

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- $\{\Omega_1, \Omega_2\}$ is $\varphi-$transversal at $\bar{x}$ with $\delta_1 := \delta$ and some $\delta_2 > 0$ if and only if

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Semitransversality $\Rightarrow$ Transversality

- If $\{\Omega_1, \Omega_2\}$ is $\varphi-$semitransversal at $\bar{x}$ with some $\delta > 0$, then it is $\psi-$transversal at $\bar{x}$ with any $\psi \in \widehat{\mathcal{C}}_\delta$, $\delta_1 := \delta$ and any $\delta_2 > 0$ such that $\delta_2 + \psi^{-1}(\delta) \leq \varphi^{-1}(\delta)$.

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- Let $q \in]0, 1]$. The collection $\{\Omega_1, \Omega_2\}$ is semitransversal of order q at $\bar{x}$ if and only if it is transversal of order q at $\bar{x}$.

Slope

X - metric space, $f : X \to \mathbb{R} \cup \{+\infty\}$, $x \in \operatorname{dom} f$.

$$|\nabla f|(x) := \limsup_{u \to x,\, u \neq x} \frac{[f(x) - f(u)]_+}{d(x,u)}, \quad |\nabla f|^{\diamond}(x) := \sup_{u \neq x} \frac{[f(x) - f_+(u)]_+}{d(x,u)}.$$

where $\alpha_+ := \max\{0, \alpha\}$.

- [1] $|\nabla f|(x)$ provides the rate of steepest descent of f at x

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$\Omega_1, \Omega_2, \varphi$ - convex, $\gamma := ((\varphi'_+(0))^{-1} + 1)^{-1}$

Theorem

If $\{\Omega_1, \Omega_2\}$ *is* $\varphi-$*semitransversal at* $\bar{x}$ *with some* $\delta > 0$*, then*

$$\limsup_{\substack{u_i \xrightarrow{\Omega_i} \bar{x}\ (i=1,2),\ u \to \bar{x} \\ (u_1,u_2,u) \neq (\bar{x},\bar{x},\bar{x})}} \frac{\varphi\left(\max\limits_{i=1,2} \|x_i\|\right) - \varphi\left(\max\limits_{i=1,2} \|u_i - x_i - u\|\right)}{\|(u_1, u_2, u) - (\bar{x}, \bar{x}, \bar{x})\|_\gamma} \geq 1$$

$\forall x_1, x_2 \in X$ *satisfying* $0 < \max_{i=1,2} \|x_i\| < \varphi^{-1}(\delta)$.

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$\forall x_1, x_2 \in X$ *satisfying* $0 < \max_{i=1,2} \|x_i\| < \varphi^{-1}(\delta)$.

- $\|(x_1, x_2, x)\|_\gamma := \max\left\{\|x\|, \gamma \max_{i=1,2} \|x_i\|\right\}$
- $f(u_1, u_2, u) := \varphi\left(\max\limits_{i=1,2} \|u_i - x_i - u\|\right) + i_{\Omega_1 \times \Omega_2}(u_1, u_2)$

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$\forall x_1, x_2 \in X$ *satisfying* $0 < \max_{i=1,2} \|x_i\| < \varphi^{-1}(\delta)$.

$$\varphi'\left(\max_{i=1,2} \|x_i\|\right) \limsup_{\substack{u_i \xrightarrow{\Omega_i} \bar{x}\ (i=1,2),\ u \to \bar{x} \\ (u_1,u_2,u) \neq (\bar{x},\bar{x},\bar{x})}} \frac{\max\limits_{i=1,2} \|x_i\| - \max\limits_{i=1,2} \|u_i - x_i - u\|}{\|(u_1, u_2, u) - (\bar{x}, \bar{x}, \bar{x})\|_\gamma} \geq 1$$

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Theorem

If $\{\Omega_1, \Omega_2\}$ *is* $\varphi-$*subtransversal at* $\bar{x}$ *with some* $\delta_1 > 0$ *and* $\delta_2 > 0$, *then*

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X - normed space, $f: X \to \mathbb{R} \cup \{+\infty\}$ - convex, $\bar{x} \in \operatorname{dom} f$.

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X - normed space, Ω - convex, $\bar{x} \in \Omega$.

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Proposition

If $\{\Omega_1, \Omega_2\}$ *is* $\varphi-$*semitransversal at* $\bar{x}$ *with some* $\delta > 0$, *then*

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- $\|(x_1^*, x_2^*, x^*)\|_\gamma = \|x^*\| + \frac{1}{\gamma}(\|x_1^*\| + \|x_2^*\|)$
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$$F(x) := (\Omega_1 - x) \times (\Omega_2 - x), \quad x \in X.$$

Note that $(\bar{x}, (0,0)) \in \operatorname{gph} F$.

- $\{\Omega_1, \Omega_2\}$ is $\varphi-$semitransversal at $\bar{x}$ if and only if F is $\varphi-$semiregular at $(\bar{x}, (0,0))$.
- $\{\Omega_1, \Omega_2\}$ is $\varphi-$subtransversal at $\bar{x}$ if and only if F is $\varphi-$subregular at $(\bar{x}, (0,0))$.
- $\{\Omega_1, \Omega_2\}$ is $\varphi-$transversal at $\bar{x}$ if and only if F is $\varphi-$regular at $(\bar{x}, (0,0))$.

Transversality & Regularity

X, Y - normed spaces, $F : X \rightrightarrows Y$, $(\bar{x}, \bar{y}) \in \operatorname{gph} F$.

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- If F is $\varphi-$semiregular at $(\bar{x}, \bar{y})$, then $\{\Omega_1, \Omega_2\}$ is $\psi-$semitransversal at $(\bar{x}, \bar{y})$.
- If F is $\varphi-$subregular at $(\bar{x}, \bar{y})$, then $\{\Omega_1, \Omega_2\}$ is $\psi-$subtransversal at $(\bar{x}, \bar{y})$.
- If F is $\varphi-$regular at $(\bar{x}, \bar{y})$, then $\{\Omega_1, \Omega_2\}$ is $\psi-$transversal at $(\bar{x}, \bar{y})$.

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Let $\psi(t) := \varphi(t/2)$.

- If $\{\Omega_1, \Omega_2\}$ is $\varphi-$semitransversal at $(\bar{x}, \bar{y})$, then F is $\psi-$semiregular at $(\bar{x}, \bar{y})$.
- If $\{\Omega_1, \Omega_2\}$ is $\varphi-$subtransversal at $(\bar{x}, \bar{y})$, then F is $\psi-$subregular at $(\bar{x}, \bar{y})$.
- If $\{\Omega_1, \Omega_2\}$ is $\varphi-$transversal at $(\bar{x}, \bar{y})$, then F is $\psi-$regular at $(\bar{x}, \bar{y})$.

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THANK
YOU!

DISCUSSION