Error Bounds Revisited[1]

Alexander Kruger
(joint work with Nguyen Duy Cuong)

Variational Analysis and Optimisation Webinar
10 March 2021

[1] Research supported by the Australian Research Council, project DP160100854

Outline

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Error Bounds

Hoffman (1952); Łojasiewicz (1959);
Robinson (1975); Ioffe (1979, 2000); Mangasarian (1985); Auslender, Crouzeix (1988); Burke, Ferris (1993); Cornejo, Jourani, Zălinescu (1997); Pang (1997); Deng (1998); Klatte (1998); Lewis, Pang (1998); Ye (1998); Bauschke, Borwein, Li (1999); Studniarski, Ward (1999); Jourani (2000); Ng, Zheng (2000, 2001, 2019); Henrion, Outrata (2001, 2005); Wu, Ye (2001, 2002, 2003); Azé, Corvellec (2002, 2004, 2014, 2017); Burke, Deng (2002, 2005); Henrion, Jourani (2002); Azé (2003); Zălinescu (2003); Bosch, Jourani, Henrion (2004); Huang, Ng (2004); Ng, Yang (2004); Ngai, Théra (2004, 2008, 2009); Corvellec, Motreanu (2008); Ioffe, Outrata (2008); Penot (2010, 2019); Meng, Yang (2012); Chao, Cheng (2014); Yao, Zheng (2016); Drusvyatskiy, Lewis (2018); Li, Meng, Yang (2018); Chao, Wang, Liang (2019); Uderzo (2019); Drusvyatskiy, Ioffe, Lewis (2021)

Error Bounds

X – metric space, $f : X \to \mathbb{R} \cup \{+\infty\}$, $\bar{x} \in X$, $\tau > 0$

Definition

f admits a τ-error bound at $\bar{x}$ if $\exists \delta \in (0,\infty]$, $\mu \in (0,\infty]$ s.t.

$$\tau d(x,[f \le 0]) \le f(x) \quad \text{for all } x \in B_\delta(\bar{x}) \cap [0 < f < \mu]$$

and either $\bar{x} \in [f \le 0]$ or $\delta = +\infty$

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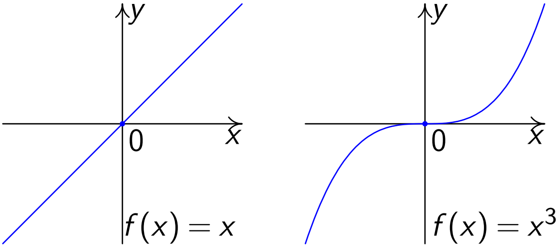

Error Bounds: Sufficient Conditions

1. Primal (slope) conditions
2. Dual (subdifferential) conditions

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(2a) $\Rightarrow$ (1b): Clarke subdifferential sum rule
(2b) $\Rightarrow$ (1b): Fréchet subdifferential sum rule

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Error Bounds

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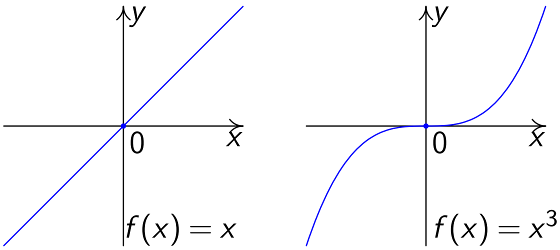

Slopes

X – metric space, $f : X \to \mathbb{R} \cup \{+\infty\}$, $x \in \operatorname{dom} f$

Slope (De Giorgi, Marino and Tosques, 1980):

$$|\nabla f|(x) := \limsup_{u \to x,\, u \neq x} \frac{[f(x) - f(u)]_+}{d(u,x)}$$

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X – normed space

Subdifferential slope (Simons, 1991): $|\partial f|(x) := d(0, \partial f(x))$

$|\partial^F f|$, $|\partial^C f|$

Slopes

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$$|\partial f|(x) := d(0, \partial f(x))$$

Lemma

Let X be a metric space, $f : X \to \mathbb{R} \cup \{+\infty\}$, and $x \in \operatorname{dom} f$

1. *If $f(x) > 0$, then* $|\nabla f|(x) \leq |\nabla f|^{\diamond}(x)$
2. *If X is Banach and f is lsc, then* $|\nabla f|(x) \geq |\partial^C f|(x)$
3. *If X is Asplund space f is lsc, then* (Ioffe, 2000)

$$|\nabla f|(x) \geq \liminf_{u \to x,\, f(u) \to f(x)} |\partial^F f|(u)$$

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$$|\mathfrak{D} f| := |\mathfrak{D} f|^{\circ} \cup \{|\nabla f|^{\diamond}\}, \quad |\mathfrak{D} f|^{\dagger} := |\mathfrak{D} f| \setminus \{|\partial^F f|\}$$

Linear Error Bounds: 'Fixed x' Statement

X – complete metric space, $f : X \to \mathbb{R} \cup \{+\infty\}$ lsc, $x \in [f > 0]$, $\tau > 0$

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Proposition

Let $\alpha \in]0,1]$. The inequality $\tau d(x, [f \leq 0]) \leq f(x)$ holds, provided that one of the following conditions is satisfied:

1. $|\breve{\nabla} f| \in |\mathfrak{D} f|^{\dagger}$ *and* $\alpha |\breve{\nabla} f|(u) \geq \tau$

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$$d(u,x) < \alpha d(x,[f \le 0]), \tag{1}$$

$$\alpha f(u) < \tau d(u,[f \le 0]), \quad f(u) < \tau d(x,[f \le 0]) \tag{2}$$

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2. X *is Asplund and* $\exists \mu > f(x)$ *s.t.* $\alpha|\partial^F f|(u) \ge \tau$ *for all* $u \in X$ *satisfying* $f(u) < \mu$ *and conditions* (1) *and* (2)

Linear Error Bounds: General Statement

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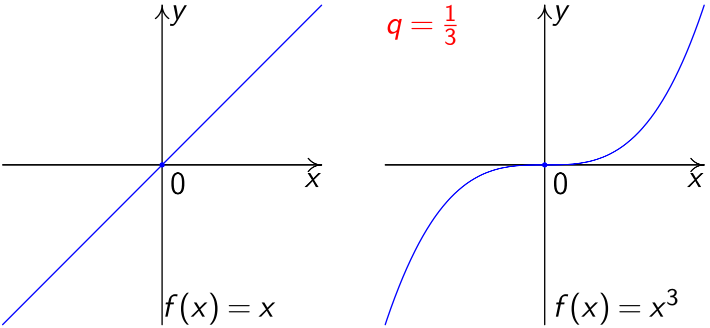
$$\max\{\alpha, 1-\alpha\} f(u) < \tau d(u, [f \le 0])$$

Outline

Nonlinear Error Bounds



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X – metric space, $f : X \to \mathbb{R} \cup \{+\infty\}$, $\bar{x} \in X$, $\varphi \in \mathscr{C}^1$

Definition

f admits a φ-error bound at $\bar{x}$ if $\exists \delta \in (0, \infty]$, $\mu \in (0, \infty]$ s.t.

$$d(x, [f \leq 0]) \leq \varphi(f(x)) \quad \text{for all } x \in B_\delta(\bar{x}) \cap [0 < f < \mu]$$

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$$\mathscr{C}^1 := \{\varphi : \mathbb{R}_+ \to \mathbb{R}_+ \mid \varphi(0) = 0,\ \varphi'(t) > 0 \text{ for all } t > 0$$
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$\varphi \circ f$ 'change-of-function' approach (Azé and Corvellec, 2017)

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If φ' *is nonincreasing, then* $|\mathfrak{D} f|^\circ$ *can be replaced with* $|\mathfrak{D} f|$

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$\alpha\varphi'(f(u))$ a variable coefficient

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for all $u \in B_\delta(\bar{x}) \cap [0 < f < \mu]$ satisfying

$$\max\{\alpha, 1-\alpha\} \varphi(f(u)) < d(u, [f \le 0])$$

Proof: $\varphi'(\varphi^{-1}((\max\{\alpha, 1-\alpha\})^{-1} d(u, [f \le 0]))) \le \varphi'(f(u))$

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Conclusions

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- Collections of slope operators and 'universal' statements

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- Collections of slope operators and ‘universal’ statements
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$$\Downarrow$$

- Linear error bounds: general statement

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Related Topics

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- Special families of functions
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References

- N. D. Cuong and A. Y. Kruger, *Error bounds revisited*, arXiv: **2012.03941** (2020)


Thank you