Circumcentering projection type methods

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joint work with

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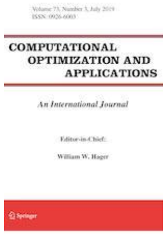
Content

Roger Behling, José Yunier Bello Cruz, Luiz-Rafael dos Santos.
Circumcentering the Douglas-Rachford method, 2018

Roger Behling, José Yunier Bello Cruz, Luiz-Rafael Santos.
On the linear convergence of the circumcentered-reflection Method, 2018

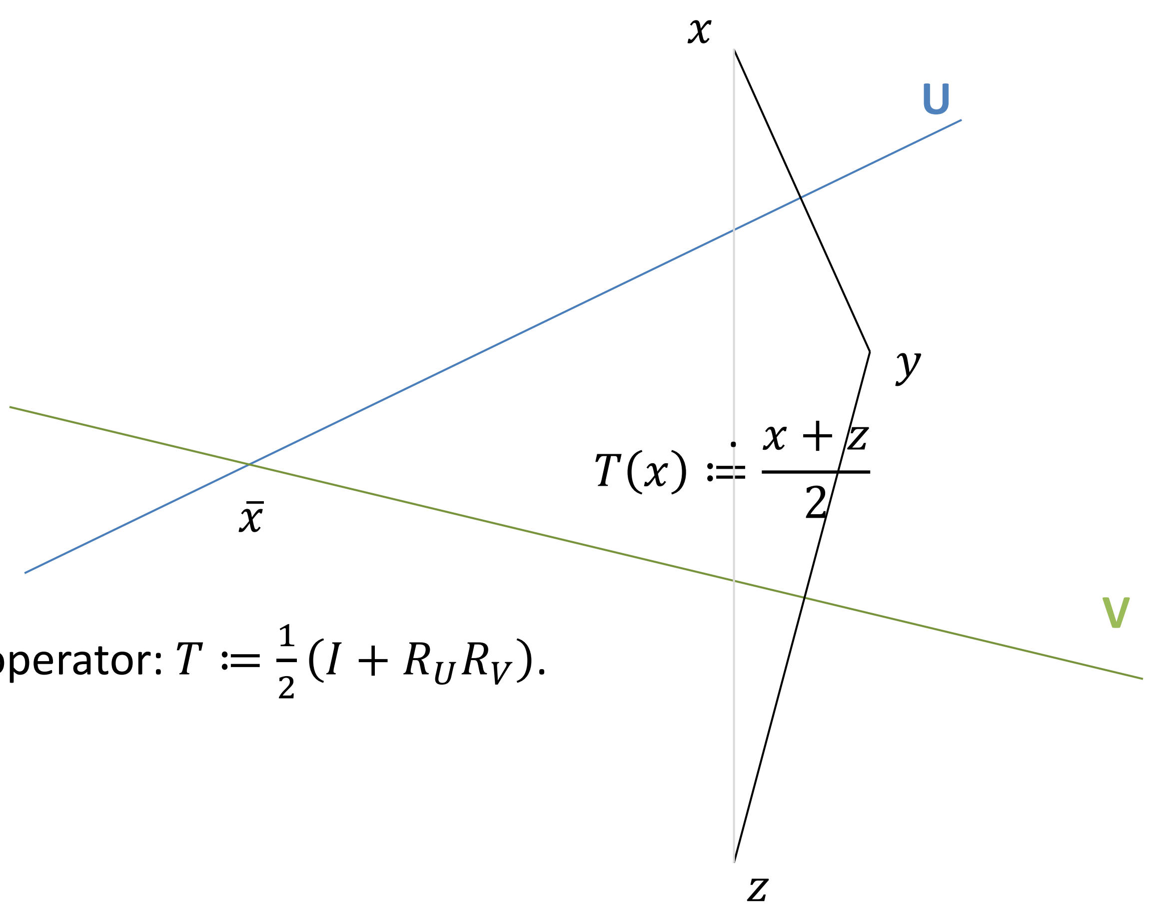
Roger Behling, José Yunier Bello Cruz, Luiz-Rafael Santos.
The Block-wise Circumcentered-Reflection Method, 2019



Roger Behling, José Yunier Bello Cruz, Luiz-Rafael Santos.
On the Circumcentered-Reflection Method for the Convex Feasibility Problem, 2020

DRM - Douglas-Rachford Method

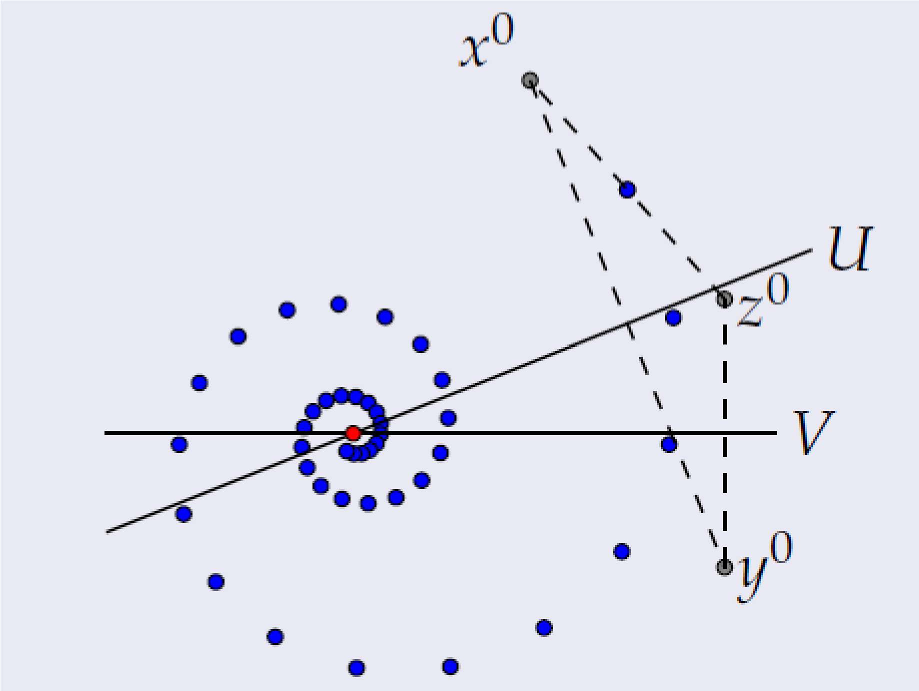
Best approximation problem: Given $x \in \mathbb{R}^n$, find $P_S(x)$, where $S \coloneqq U \cap V$, and U, V are affine subspaces with nonempty intersection.



Douglas-Rachford operator: $T \coloneqq \frac{1}{2}(I + R_U R_V)$.

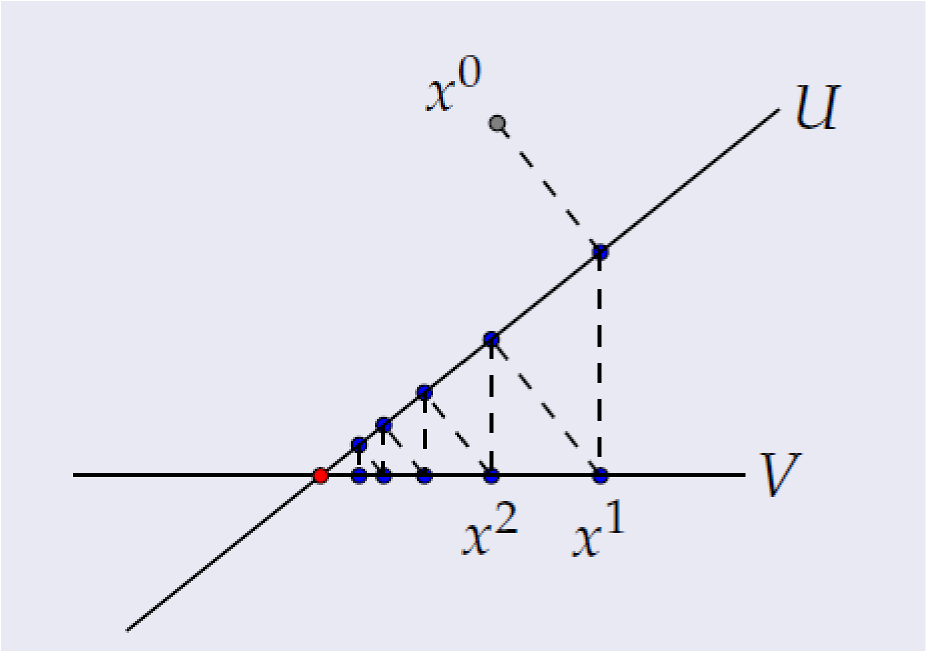
Remark: $T(x)$ is also the Cimmino iteration calculated at y

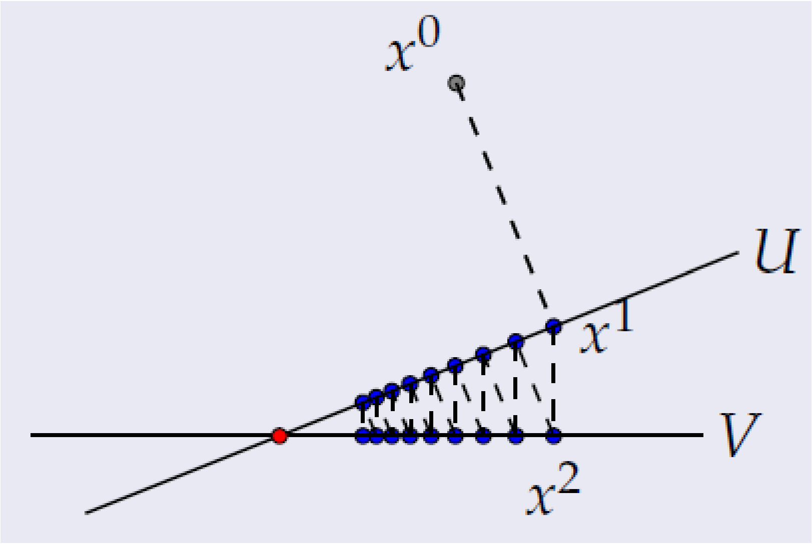
DRM - Douglas-Rachford Method



DRM can be seen as ADMM via duality

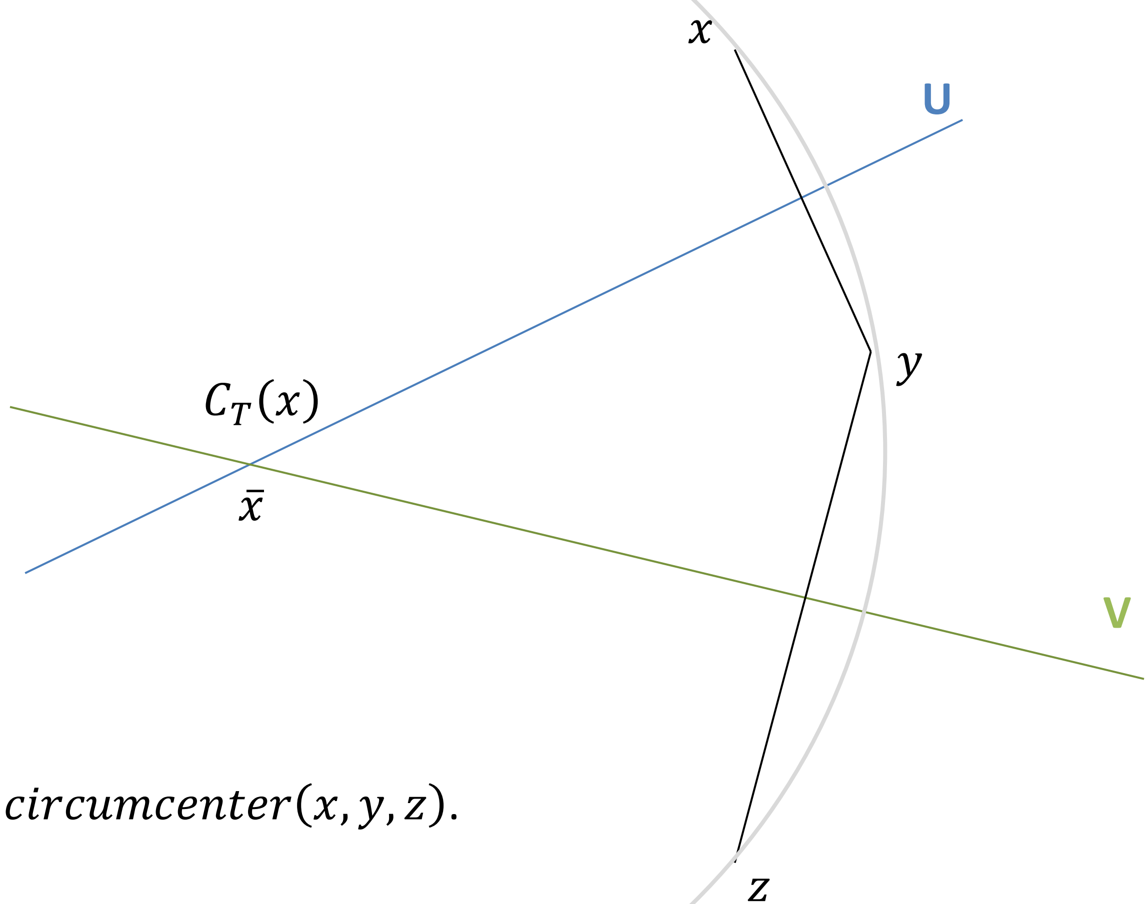
MAP – Method of Alternating Projections





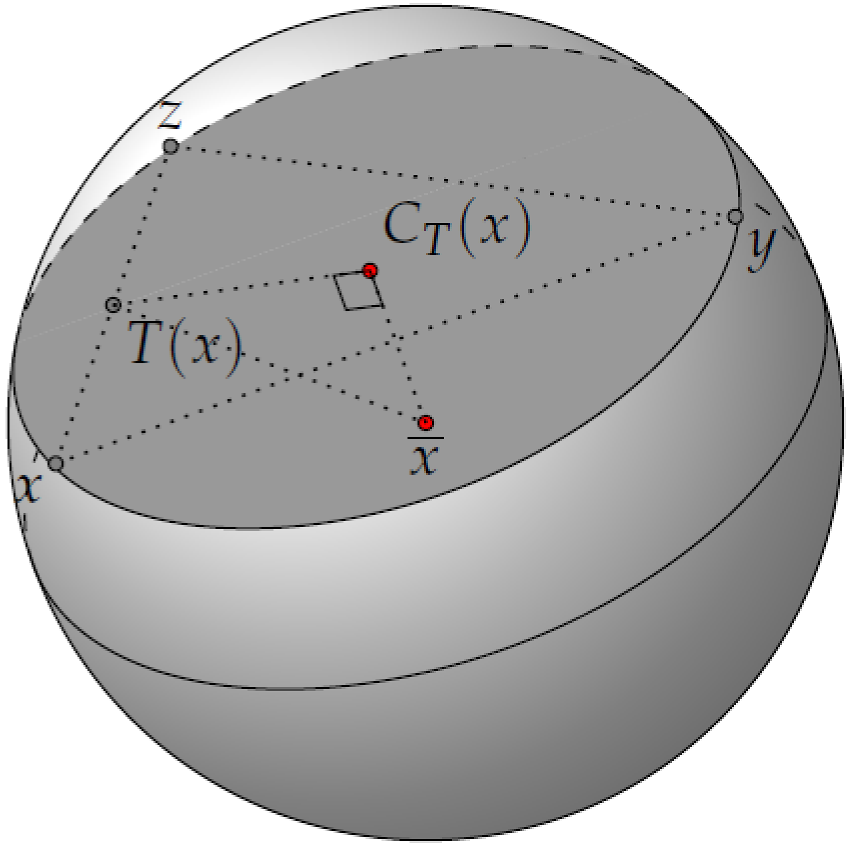
CRM – Circumcentered-reflection method

Problem: find $\bar{x} := P_S(x)$ with $S := U \cap V$



$C_T(x) := circumcenter(x, y, z).$

Geometric interpretation of $T(x)$ and $C_T(x)$



Definition of $C_T(x)$:

(i) $C_T(x)$ belongs to the affine subspace determined by the points $x, y \coloneqq R_U(x), z \coloneqq R_V R_U(x)$;

(ii) $C_T(x)$ is equidistant to the points $x, y \coloneqq R_U(x), z \coloneqq R_V R_U(x)$.

Convergence analysis for CRM

Lemma: Let $x \in \mathbb{R}^n$. Then, the projection $P_{U\cap V}(x)$ onto the affine subspace defined by the points $x, y \coloneqq R_U(x), z \coloneqq R_V R_U(x)$ is given by $C_T(x)$.

Consequence: $\|C_T(x) - P_{U\cap V}(x)\| \leq \|T(x) - P_{U\cap V}(x)\|$ for all $x \in \mathbb{R}^n$.

Theorem: Let $x \in \mathbb{R}^n$. Then, the sequence $\{C_T^k(P_U(x))\}$ converges linearly to $P_{U\cap V}(x)$ and the rate is at least $c_F \in [0,1)$, the cosine of the Friedrichs angle between U and V.

Remarks: Remind that

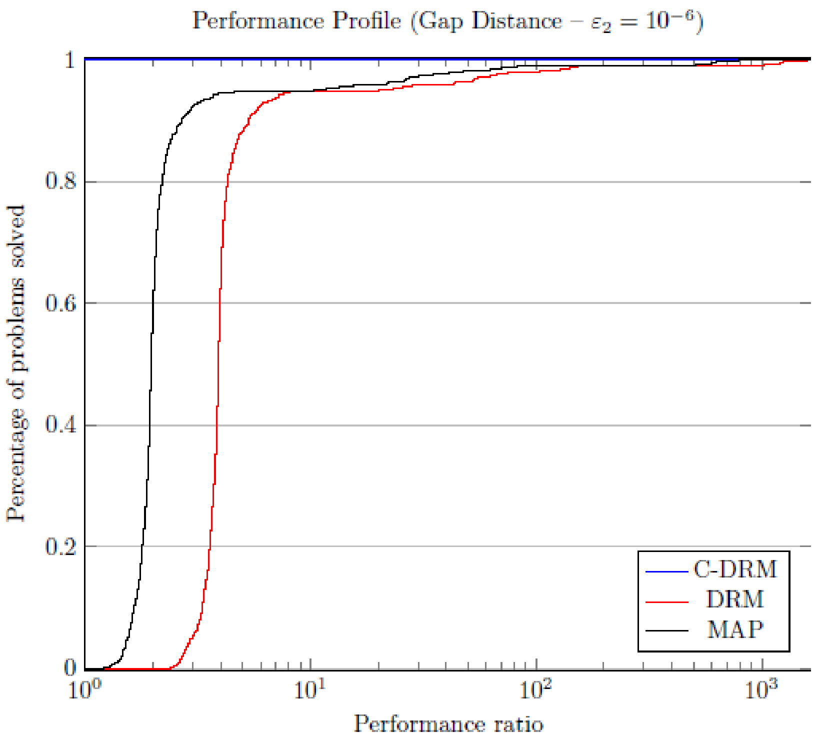
$$c_F \coloneqq \sup\{u^T v | u \in (\widehat{U} \cap \widehat{V}) \cap \widehat{U}^\perp, v \in (\widehat{U} \cap \widehat{V}) \cap \widehat{V}^\perp, \|u\| = 1, \|v\| = 1\},$$

where $\widehat{U}, \widehat{V}$ are subspaces and translations of U, V, respectively.

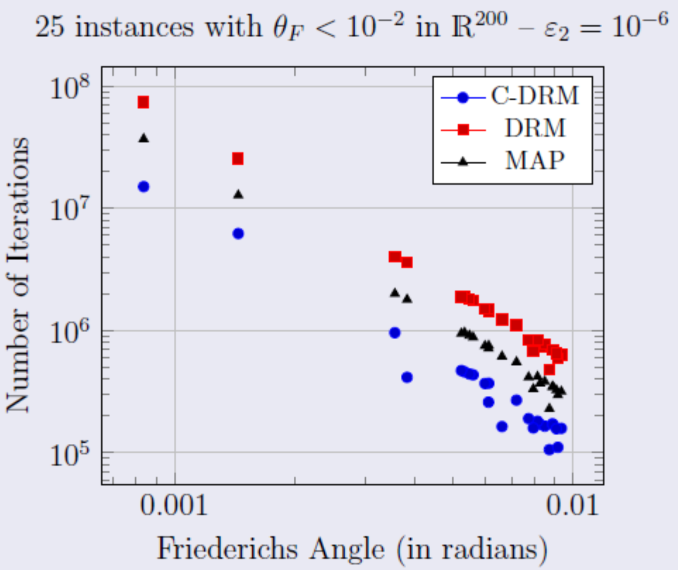
(i) c_F is the sharp rate of the original de Douglas-Rachford method;

(ii) question: does CDRM converge with linear rate strictly better than c_F?

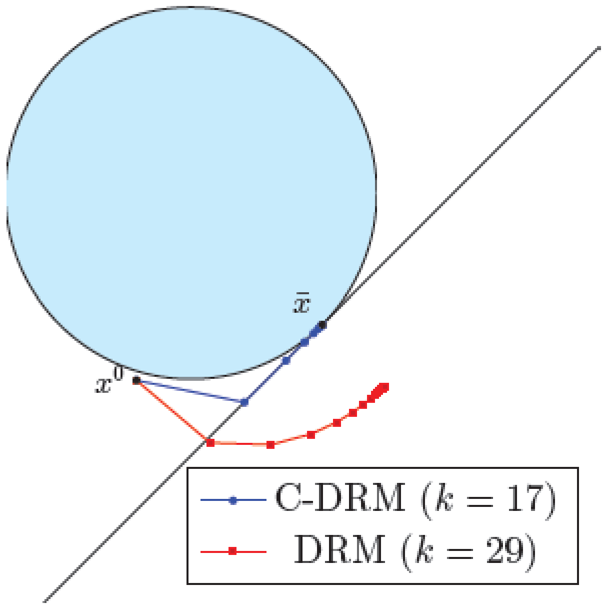
Numerical experiments

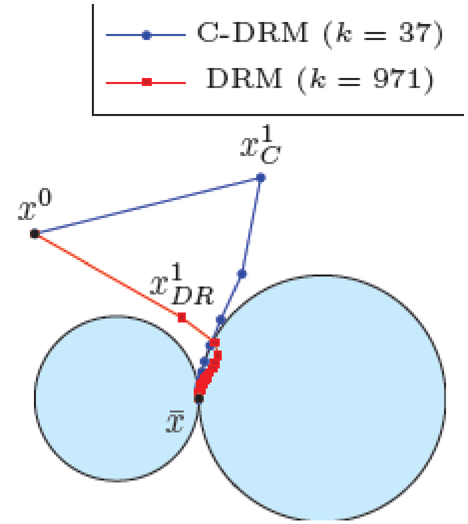


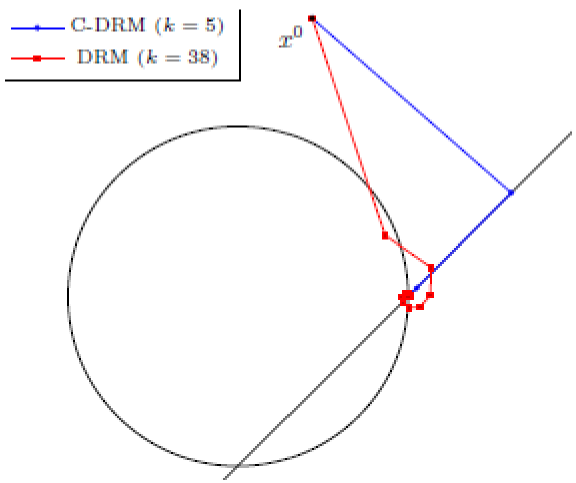
Small values of θ_F

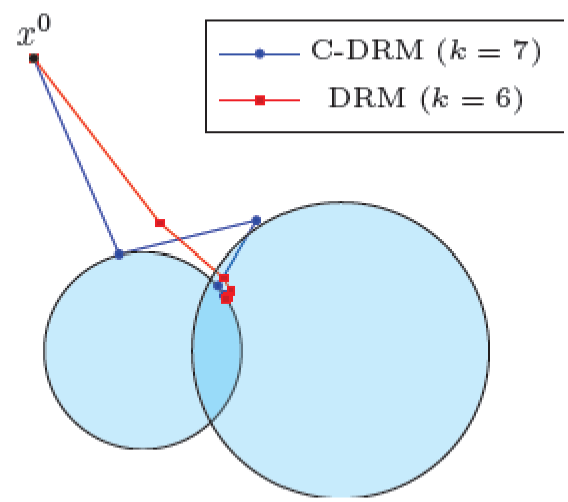


Non-affine examples









The many set case (still affine)

Best approximation problem: Given $x \in \mathbb{R}^n$, find $P_S(x)$, where $\mathrm{S} \coloneqq U_1 \cap U_2 \cap \cdots U_m$, with U_i's being affine subspaces and S is nonempty.

“Game rules”: we can use projections and/or reflections onto the U_i's.

The pure DRM may fail for $m > 2$

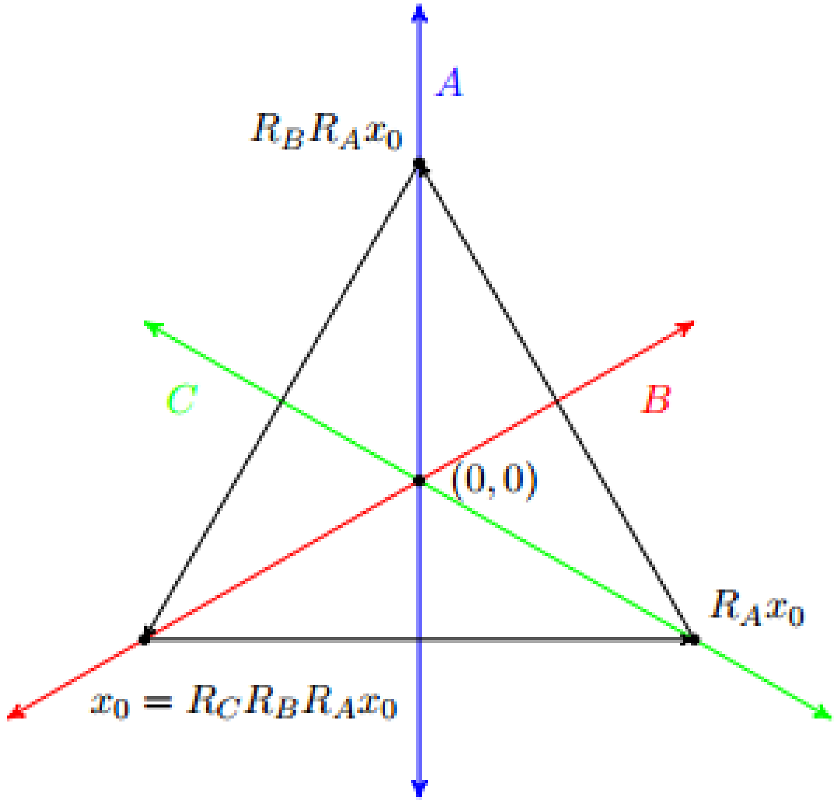


Figure by Tam and Borwein

Our ideia: Auxiliar Operator

Define $A: \mathbb{R}^n \to \mathbb{R}^n$ based on

$$A_1 := \frac{1}{2}\left(Id + P_{U_1}\right)$$

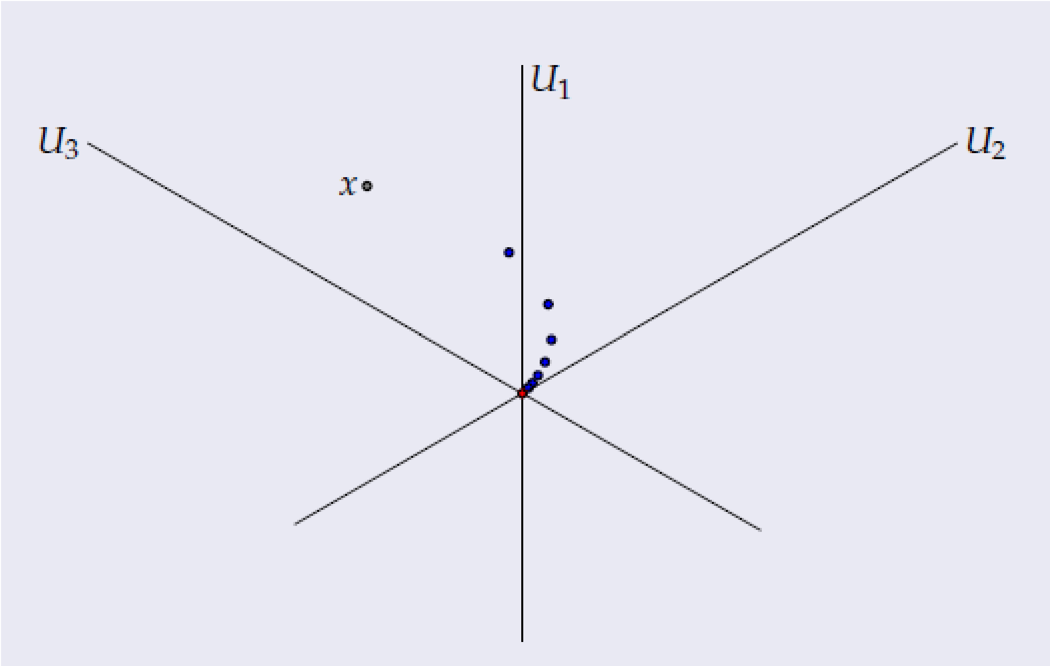
$$A_2 := \frac{1}{2}\left(Id + P_{U_2}R_{U_1}\right)$$

$$A_3 := \frac{1}{2}\left(Id + P_{U_3}R_{U_2}R_{U_1}\right)$$

$\vdots$

$$A_m := \frac{1}{2}\left(Id + P_{U_m}R_{U_{m-1}}R_{U_2}R_{U_1}\right)$$

Let $A := \frac{1}{m}(A_1 + A_2 + \cdots + A_m)$



Results: A is firmly nonexpansive, $Fix_A = S$ and for any $x \in \mathbb{R}^n$, $\{A^k(x)\}$ converges linearly to $P_S(x)$.

CRM for m affine subspaces

Definition of $C(x)$:
(i) $C(x)$ belongs to the affine subspace W_x defined by the $m+1$ vectors
$x, R_{U_1}(x), R_{U_2}R_{U_1}(x), R_{U_3}R_{U_2}R_{U_1}(x), \dots, R_{U_m} \dots R_{U_1}(x)$;
(ii) $C(x)$ is equidistant to
$x, R_{U_1}(x), R_{U_2}R_{U_1}(x), R_{U_3}R_{U_2}R_{U_1}(x), \dots, R_{U_m} \dots R_{U_1}(x)$.

Lemma: Let $x \in \mathbb{R}^n$. Then, the projection $P_S(x)$ onto the affine subspace W_x is given by $C(x)$.

Consequence: $\|C(x) - P_S(x)\| \leq \|A(x) - P_S(x)\|$ for all $x \in \mathbb{R}^n$.

Theorem: Let $x \in \mathbb{R}^n$ and $\mathrm{S} \coloneqq U_1 \cap U_2 \cap \cdots U_m$, with all U_i's being affine subspaces and S non-empty. Then, the sequence $\{C^k(x)\}$ converges linearly to $P_S(x)$.

Computation of a circumcenter

Consider the notation $x^{(i)} = R_{U_i} \dots R_{U_2} R_{U_1}(x)$, for $i = 1, \dots, m$. We want two things:

First (Equidistance):

$$P_{span\{x^{(i)}-x\}}(C(x) - x) = \frac{1}{2}\left(x^{(i)} - x\right) \text{ for each } i = 1, \dots, m;$$

Second (Being in affine$\left(x, x^{(1)}, x^{(2)}, \dots, x^{(m)}\right)$)

$$C(x) - x = \sum_{j=1}^{m} \alpha_i \left(x^{(j)} - x\right)$$

This yields the solvable $m \times m$ linear system in $\alpha \in \mathbb{R}^m$ whose $i - th$ row reads as

$$\sum_{j=1}^{m} \alpha_j \left\langle x^{(j)} - x, x^{(i)} - x \right\rangle = \frac{1}{2} \left\| x^{(i)} - x \right\|^2$$

$C(x)$ outcomes univocally from this linear system. Uniqueness in α, however, depends on linear independence of the vectors $x^{(i)} - x$, which is not always the case.

Block-wise CRM for m affine subspaces

Example: Let $m = 7$, i.e., $S = \bigcap_1^7 U_i$. Take, for instance, the following blocks of affine subspaces $B_1 \coloneqq \{U_1, U_2\}$, $B_2 \coloneqq \{U_3, U_4, U_5, U_6\}$, $B_3 \coloneqq \{U_7\}$.

Block-wise CRM:

For a given x^k, we define $x^{k+1} = C_{Bw-CRM}(x^k) \coloneqq C_{B_3}\left(C_{B_2}\left(C_{B_1}(x^k)\right)\right)$

Theorem: Let $x \in \mathbb{R}^n$. Then, the sequence $\{C_{Bw-CRM}^k(x)\}$ converges linearly to $P_S(x)$.

Remarks:

- MAP (method of alternating projections) is a Bw-CRM where all blocks contain exactly one affine subspace.
- Bw-CRM with one full block (original CRM) solves hyperplane intersection problems in one single step.

Experiments on the Block-wise CRM

Bw-CRM applied to CT – Matrix size: 5732 × 2500 – Budget of 10 iterations.

(a) Exact Shepp-Logan (b) Bw-CRM-1 (MAP) (c) Bw-CRM-16

(d) Bw-CRM-64 (e) Bw-CRM-256

General convex inclusions

$$Find\ x^* \in X := \bigcap_{i=1}^{m} X_i$$

Where X_i is closed and convex for all $i = 1, \dots, m$. We assume also that X is nonempty and that the orthogonal projetions onto each X_i are computable.

Pierra's product space reformulation:

Let $W := X_1 \times X_2 \times \cdots \times X_m$ and $D := \{(x, x, \dots, x) \in \mathbb{R}^{nm} | x \in \mathbb{R}^n\}$. Then, finding $x^* \in X$ is equivalent to solving the following problem

$$Find\ z^* \in W \cap D$$

CRM for product space reformulation

Consider

$$Find\ z^* \in K \cap U,$$

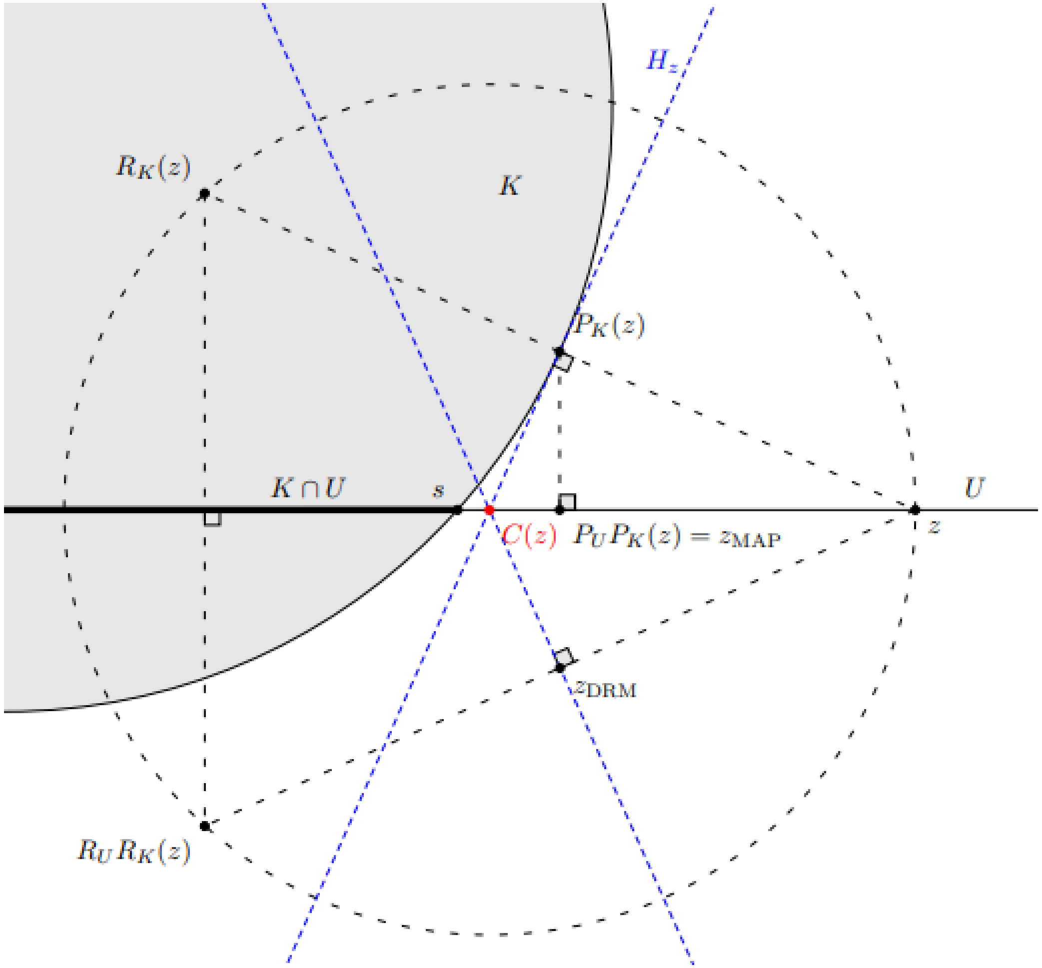
with K closed and convex and U an affine subspace. Assume also that their intersection is nonempty.

Theorem: Let $z^0 \in U$ be given and consider the sequence $\{z^k\}$ generated by

$$z^{k+1} := circumcenter\{z^k, R_K(z^k), R_U R_K(z^k)\}.$$

Then, $\{z^k\}$ converges to a point in $K \cap U$. Moreover, each z^{k+1} is closer to $K \cap U$ than the MAP and DR points calculated at z^k.

Geometry of CRM



Numerical experiments

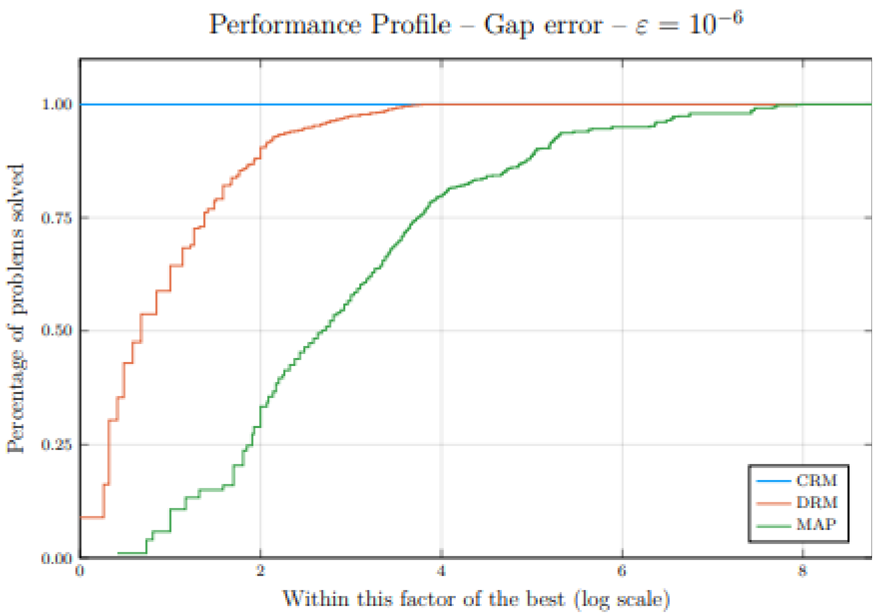


Fig. 2: Experiments with affine subspaces and the second order cone.

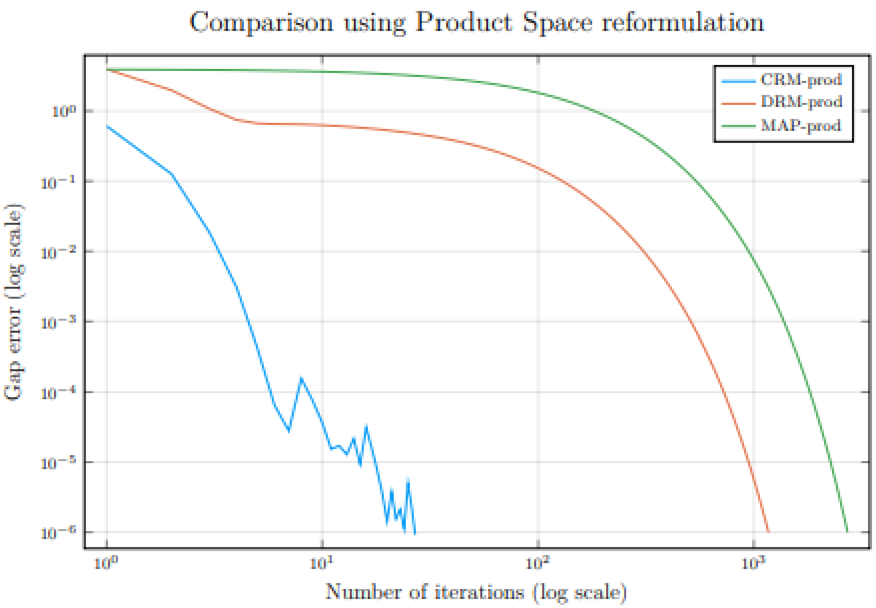


Fig. 3: Polyhedral feasibility using the product space reformulation.

	mean	min	median	max
CRM	4.727	3	5.0	6
DRM	11.602	4	8.0	83
MAP	83.981	4	32.0	1063

	mean	min	median	max
CRM	41.5	19.0	38.0	89.0
DRM	1441.15	1036.0	1470.5	1586.0
MAP	2768.3	2534.0	2787.0	2952.0

New work and ideas in progress

- *The circumcentered-reflection method achieves better rates than alternating projections*. R. Arefidamghani, R. Behling, Y. Bello-Cruz, A. Iusem, LR Santos (accepted in COAP 2021)
- *Circumcentering outer-approximate reflections* G. Araújo, R. Arefidamghani, R. Behling, Y. Bello-Cruz, A. Iusem, LR Santos (to be submitted soon)

Future research

- Investigation on the suitability of generalized circumcenters for *basis pursuit*, *sparse affine feasibility problems, superlinear convergence of CRM and* [content hidden].

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THANK YOU VERY MUCH FOR YOUR ATTENTION