Smooth approximation of d.c.functions

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Variational Analysis and Optimisation Webinars

Let us formulate some definitions and statements of Convex Analysis.
A set $X \subset \mathbb{R}^n$ is called convex, if for all $x_1 \in X$ and $x_2 \in X$ the next formula

$$\lambda x_1 + (1 - \lambda)x_2 \in X \quad \forall \lambda \in [0, 1] \subset \mathbb{R},$$

hold.
We assume that the empty set $\emptyset$ is convex by definition.
The sum of two convex sets $X_1, X_2 \subset \mathbb{R}^n$ is called the set

$$X = X_1 + X_2 = \{x_1 + x_2 \mid x_1 \in X_1, \quad x_2 \in X_2\}.$$

Sometimes the set $X = X_1 + X_2$ is called the algebraic sum of two convex sets X_1 and X_2 or the Minkowski sum.
By writing $X_1 - X_2$ we will understand the set $X_1 + (-X_2)$.
If the set $X \subset \mathbb{R}^n$ is convex and $\alpha X, \alpha \in \mathbb{R}$, then

$$\alpha X = \{y \in \mathbb{R}^n \mid y = \alpha x,\ x \in X\}.$$

Let $K \subset \mathbb{R}^n$ be a convex cone.

$$K^* = \{g \in \mathbb{R}^n \mid \langle g, x\rangle \geqslant 0 \quad \forall x \in K\}$$

is called the conjugate cone to K.
Let $X \subset \mathbb{R}^n$ be a closed and convex set, a point $x \in X$.
A set

$$N(X, x) = \left\{g \in \mathbb{R}^n \;\middle|\; \langle g, z - x\rangle \leqslant 0 \quad \forall z \in X \right\}$$

is called the normal cone to the set X at x.
The normal cone is a closed convex cone.

A closed convex set is called smooth if for each boundary point there is only one support hyperplane.

Theorem 1.

If at every boundary point of a closed convex set a normal cone consists of a single ray then this set is smooth.

This is an obvious property of a smooth set.

Let a set $X \subset \mathbb{R}^n$ be closed and convex and X does not coincide with $\mathbb{R}^n$.
Fix $\varepsilon > 0$. Form a closed convex set

$$X(\varepsilon) = X + \varepsilon B_1(0_n),$$

where

$$B_r(a) = \{x \in \mathbb{R}^n \mid ||x - a|| \leq r\},$$

$||x|| = \sqrt{\langle x, x \rangle}$.

At this picture you can see that the rectangle vertices are smoothed out.

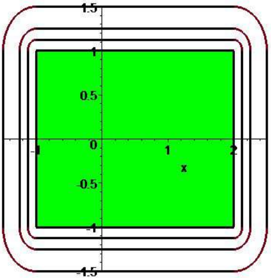

Fig. 1. The family of sets X_ε.

Theorem 2.

The following statements take place.

- For every $\varepsilon > 0$ a normal cone to the set $X(\varepsilon)$ consists of a single ray at each boundary point $z_0 \in \text{bd}\,(X(\varepsilon))$.
- For every $\varepsilon > 0$ the set $X(\varepsilon)$ is smooth.
- The next inclusion

$$N(X(\varepsilon), z_0) \subset N(X, x_0)$$

holds, where $x_0 = \arg\min_{x \in X} ||x - z_0||$.

Let $X \subset \mathbb{R}^n$ be an unbounded closed convex set.
Consider a multivalued mapping

$$X(\cdot) : (0, +\infty) \to 2^{\mathbb{R}^n}.$$

Theorem 3.

This multivalued mapping is Kuratowski continuous.

Theorem 4.

Let $X \subset \mathbb{R}^n$ be a compact convex set. Then

$$\rho_H(X(\varepsilon), X) \to 0, \text{if } \varepsilon \to +0,$$

where $\rho_H(X(\varepsilon), X)$ is the Hausdorff metric,

$$\rho_H(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B} ||a-b||, \sup_{b\in B}\inf_{a\in A} ||a-b||\right\}.$$

Convex functions

Let $f : \mathbb{R}^n \to \mathbb{R} \bigcup \{+\infty\} \bigcup \{-\infty\}$.
The set

$$\text{dom } f = \{x \in \mathbb{R}^n \mid \; f(x) < +\infty\}.$$

is called the effective domain of a function f.
The set

$$\text{epi } f = \{[x, \mu] \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leqslant \mu\}$$

is called the epigraph of f.

A function f is called a convex if epi f is convex. A convex function f is said to be proper if its epigraph is non-empty and contains no vertical lines.

For proper convex functions it is possible to give another definition which equivalent to the above.

A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called convex if

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leqslant \lambda_1 f(x_1) + \lambda_2 f(x_2) \quad \forall x_1, x_2 \in \mathbb{R}^n,$$

$$\lambda_1, \lambda_2 \geq 0, \; \lambda_1 + \lambda_2 = 1.$$

Let f be a convex function.

If $x \in \text{int dom} f$, then f is continuous at x.

Let f be a convex function. If the partial derivatives of f with respect to each variable exist at a point $x \in \text{int dom} f$ then f is differentiable at x.

A proper convex function is called essentially smooth if it satisfies the following three conditions:

- The set $X = \text{int (dom f)}$ is not empty;
- f is differentiable at each $x \in X$;
- if $x_1, x_2, \ldots$ is a sequence in X converging to a boundary point x of X, then

$$\lim_{i \to +\infty} |f'(x_i)| = +\infty.$$

Any smooth convex function on $\mathbb{R}^n$ will be essentially smooth, as the set of sequences satisfying the last condition is empty.

The conjugate of a function f is

$$f^*(v) = \sup_{x \in \mathbb{R}^n} \{\langle x, v\rangle - f(x)\}, \;\; v \in \mathbb{R}^n.$$

Obviously, that the equality

$$f^*(v) = \sup_{x \in \text{dom } f} \{\langle x, v\rangle - f(x)\}, \;\; v \in \mathbb{R}^n,$$

is true.

Note some properties of conjugate functions.

- f^* is closed and convex (even when f is not).
- The Fenchel inequality is true:

$$f(x) + f^*(v) \geq \langle x, v \rangle \ \forall x \in \mathbb{R}^n, \ \forall v \in \mathbb{R}^n.$$

- A proper convex closed function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex if and only if when

$$f(x) = (f^*)^*(x), \quad x \in \mathbb{R}^n.$$

In this case $\mathrm{dom} f^* \neq \emptyset$.

A convex function may be nonsmooth.
Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be convex and $x \in \operatorname{dom} f$.
The set

$$\partial f(x) = \{v \in \mathbb{R}^n \mid \; f(z) - f(x) \geq \langle v, z - x\rangle \quad \forall z \in \mathbb{R}^n\}$$

is called the subdifferential of f at x.
A vector $v \in \partial f(x)$ is called a subgradient of f at x.
The subdifferential generalizes the derivative to functions which are not differentiable. If f is convex and differentiable, then its gradient at x is a subgradient.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper convex function, $x_0 \in \text{dom } f$, $\partial f(x_0) \neq \emptyset$. Then $\partial f(x_0)$ is convex and closed.

For a convex function there is a close connection between the subdifferential and its directional derivative.

If a point $x_0 \in$ int (dom f), the set $\partial f(x_0)$ is bounded, then in this case $f'(x, g)$ is finite for each $g \in \mathbb{R}^n$ and

$$f'(x_0, g) = \max_{v \in \partial f(x_0)} \langle v, g \rangle.$$

Let f be a finite convex function on $\mathbb{R}^n$. Then a multivalued mapping

$$\partial f : \mathbb{R}^n \to 2^{\mathbb{R}^n}$$

is upper semicontinuous.

Here $2^{\mathbb{R}^n}$ is a family of subsets of $\mathbb{R}^n$.

A point $x^* \in \mathbb{R}^n$ is a minimizer of a convex function f if and only if f is subdifferentiable at x and

$$0 \in \partial f(x^*)$$

Any local minimum of convex function is also a global minimum of it.

Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex functions. A function

$$f(x) = \inf_{\substack{x_1 + x_2 = x \\ x_1, x_2 \in \mathbb{R}^n}} \{f_1(x_1) + f_2(x_2)\} = \inf_{x_1 \in \mathbb{R}^n} \{f_1(x_1) + f_2(x - x_1)\}$$

is called the infimal convolution of two functions f_1, f_2 and is denoted by

$$f(x) = (f_1 \oplus f_2)(x).$$

The infimal convolution f is convex on $\mathbb{R}^n$.

The infimal convolution can also be defined in terms of addition of epigraphs f_1 and f_2:

$$(f_1 \oplus f_2)(x) = \inf \left\{ \mu \in \mathbb{R} \;\middle|\; (x, \mu) \in [\, \text{epi } f_1 + \text{epi } f_2 \,] \right\}.$$

Note some properties of convex functions obtained in the result of the infimal convolution. Let f_1 and f_2 be convex functions, then

$$\text{dom } (f_1 \oplus f_2) = \text{dom } f_1 + \text{dom } f_2.$$

Let f_1 and f_2 be closed convex functions in $\mathbb{R}^n$. Then

$$(f_1 \oplus f_2)^* = f_1^* + f_2^*. \tag{1}$$

If $\text{ri } (\text{dom } f_1) \cap \text{ri } (\text{dom } f_2) \neq \emptyset$, then

$$(f_1 + f_2)^* = f_1^* \oplus f_2^*.$$

Let f_1 and f_2 be closed convex functions on $\mathbb{R}^n$, and

$$\text{ri }(\text{dom } f_1) \cap \text{ri }(\text{dom } f_2) \neq \emptyset,$$

then, if f_1 is essentially smooth, then $f_1 \oplus f_2$ is also essentially smooth.

If functions f_1 and f_2 is not identically equal $+\infty$ and the infimal convolution $f_1 \oplus f_2$ is exact at a point $x = x_1 + x_2$, then

$$\partial(f_1 \oplus f_2)(x) = \partial f_1(x_1) \cap \partial f_2(x_2).$$

Let f_1 be a continuous convex function on $\mathbb{R}^n$ and $f_2(x) = \frac{1}{2}\langle Mx, x\rangle$, where M be a definite positive matrix. The function

$$f(x) = (f_1 \oplus f_2)(x) = \inf_{y\in\mathbb{R}^n} \left\{ f_1(y) + \frac{1}{2}\langle M(x-y), (x-y)\rangle \right\}.$$

is called the Moreau-Yosida regularization.

Example 1.

Let $X_1 \subset \mathbb{R}^n$ and $X_2 \subset \mathbb{R}^n$ be convex sets and $f_1(x) = \delta(X_1, x), f_2(x) = \delta(X_2, x)$ be the indicator functions of X_1 and X_2, then

$$f(x) = (f_1 \oplus f_2)(x) = \delta(X_1 + X_2)(x).$$

For example, if

$$X_1 = \text{co}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right\} \subset \mathbb{R}^2,\ X_2 = \text{co}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\} \subset \mathbb{R}^2,$$

then

$$\delta(X_1,x)=\begin{cases}0, & x\in X_1,\\ +\infty, & x\notin X_1,\end{cases}\quad \delta(X_2,x)=\begin{cases}0, & x\in X_2,\\ +\infty, & x\notin X_2,\end{cases}$$

$$X_1+X_2=\text{co}\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}-1\\-1\end{pmatrix}\right\}\subset\mathbb{R}^2,$$

$$\delta(X_1+X_2,x)=\begin{cases}0, & x\in X_1+X_2,\\ +\infty, & x\notin X_1+X_2.\end{cases}$$

Example 2.

Let $X \subset \mathbb{R}^n$ be a convex set, $f_1(x) = \delta(X, x)$ be the indicator function, f_2 be a convex function, then

$$f(x) = (f_1 \oplus f_2)(x) = \inf_{x_1 \in X} f_2(x - x_1).$$

Let

$$X = \operatorname{co}\{-1, 1\} \subset \mathbb{R}, \quad f_1(x) = \delta(X, x), \quad f_2(x) = x^2, \ x \in \mathbb{R}.$$

Then

$$f(x) = (f_1 \oplus f_2)(x) = \begin{cases} (1+x)^2, & x < -1, \\ 0, & |x| \leqslant 1, \\ (1-x)^2, & x > 1. \end{cases}$$

Fix $\varepsilon > 0$. Denote a function

$$t_\varepsilon(x) = \begin{cases} -\sqrt{\varepsilon^2 - \langle x, x\rangle}, & ||x|| \leqslant \varepsilon, \\ +\infty, & ||x|| > \varepsilon, \end{cases} \qquad x \in \mathbb{R}^n.$$

We have

$$t^*_\varepsilon(v) = \varepsilon\sqrt{1 + \langle v, v\rangle}, \quad v \in \mathbb{R}^n, \quad \varepsilon > 0.$$

The function t_ε is determined only in a ball of radius ε with the center at the zero point.

This function is essentially smooth, i.e., it is differentiable in each internal point $x \in \text{int dom } t_\varepsilon$, and, if $x_1, x_2, \ldots$ is a sequence of elements from int dom t_ε converging to the point $x \notin \text{int dom } t_{2\varepsilon}$, then

$$\lim_{i \to +\infty} |f'(x_i)| = +\infty.$$

.

Therefore, the effective domain of the function t_ε^* is the whole space $\mathbb{R}^n$.

Smooth approximation of convex functions

Consider a convex function $f : \mathbb{R}^n \to \mathbb{R}$ and a closed convex set $D \subset \mathbb{R}^n$. Denote by

$$X = \text{epi } f = \left\{ [x, \mu] \in \mathbb{R}^n \times \mathbb{R} \;\middle|\; \mu \geq f(x), \quad x \in D \right\}.$$

Construct a family of convex closed sets $Z_\varepsilon \subset \mathbb{R}^{n+1}$

$$Z_\varepsilon = X + \varepsilon B_1(0_{n+1}) \subset \mathbb{R}^{n+1}, \quad \varepsilon > 0,$$

a family of convex closed sets $D_\varepsilon \subset \mathbb{R}^n$

$$D_\varepsilon = D + \varepsilon B_1(0_n) \subset \mathbb{R}^n,$$

and a family of convex functions

$$f_\varepsilon(x) = \begin{cases} \inf \mu, & [x, \mu] \in Z_\varepsilon \\ +\infty, & \text{at other cases.} \end{cases}$$

Here

$$B_\varepsilon(0_n) = \{x \in \mathbb{R}^n \mid ||x|| \leqslant \varepsilon\}, \quad B_\varepsilon(0_{n+1}) = \{x \in \mathbb{R}^{n+1} \mid ||x|| \leqslant \varepsilon\}.$$

We have

$$\text{dom } f_\varepsilon = D_\varepsilon,$$

and for each fixed $\varepsilon > 0$ the graph of the function f_ε is the lower envelope of the corresponding sets X_ε.

Fix a positive number $\varepsilon > 0$. Let $z \in D$. Consider a family of convex functions $\{\varphi_\varepsilon(x, z)\}$

$$\varphi_\varepsilon(x, z) = f(z) + t_\varepsilon(x, z),$$

where

$$t_\varepsilon(x, z) = \begin{cases} -\sqrt{\varepsilon^2 - ||x - z||^2}, & x \in B_\varepsilon(z), \\ +\infty, & \text{at other cases.} \end{cases}$$

Here

$$B_\varepsilon(z) = \epsilon B_1(z) = \{x \in \mathbb{R}^n \mid \; ||x - z|| \leqslant \varepsilon \; \} \subset D_\varepsilon.$$

It is obvious that

$$\text{dom } \varphi_\varepsilon(\cdot, z) = B_\varepsilon(z), \quad \bigcup_{z \in D} B_\varepsilon(z) = D_\varepsilon.$$

Denote $H_\varepsilon(z) = \text{epi } \varphi_\varepsilon(\cdot, z) \subset \mathbb{R}^{n+1}$. Consider also functions

$$\varphi_\varepsilon(x) = \inf_{z \in D} \varphi_\varepsilon(x, z)$$

and its epigraphs $H_\varepsilon = \text{epi } \varphi_\varepsilon$.
From the construction of the function f_ε we have

Lemma 1.

The next equality

$$f_\varepsilon(x) = (f \oplus t_\varepsilon)(x)$$

holds where

$$t_\varepsilon(x) = \begin{cases} -\sqrt{\varepsilon^2 - ||x||^2}, & ||x|| \leqslant \varepsilon, \\ +\infty, & \text{at other points.} \end{cases}$$

Note the fact that the function t_ε is an essentially smooth function for every fixed positive ε.
Consider the function $f_\varepsilon(x) = (f \oplus t_\varepsilon)(x)$.
As the function f_ε is a closed proper convex function then the Fenchel inequality

$$f_\varepsilon^*(v) = f^*(v) + t_\varepsilon^*(v), \quad v \in \mathbb{R}^n$$

holds.

Example 3.

Let

$$f(x) = \max\left\{-2x - 6, -\frac{1}{2}x - 3, 2x - 8\right\}, \quad x \in \mathbb{R}.$$

Or

$$f(x) = \begin{cases} -2x - 6 & x \in (-\infty, -2), \\ -\frac{1}{2}x - 3 & x \in [-2, 2), \\ 2x - 8, & x \in [2, +\infty). \end{cases}$$

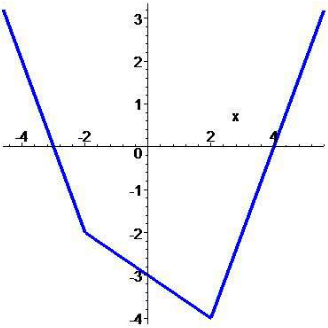

Fig.2. The functions $f(x)$.

Consider two variants.
1. Let the set D be the Euclidean space $\mathbb{R}$. Then the set of minimizers of this function consists of a single point $x^* = 2$ and $f(2) = -4$. Fix an arbitrary positive $\varepsilon > 0$. Then (see Fig.1)

$$
f_\varepsilon(x) = \begin{cases}
-2x - 6 - \sqrt{5}\varepsilon, & x \in \left(-\infty, -2 - \dfrac{2\sqrt{5}\varepsilon}{5}\right), \\
-2 - \sqrt{\varepsilon^2 - (x+2)^2}, & x \in \left[-2 - \dfrac{2\sqrt{5}\varepsilon}{5}, -2 - \dfrac{\sqrt{5}\varepsilon}{5}\right), \\
-\dfrac{1}{2}x - 3 - \dfrac{\sqrt{5}\varepsilon}{2}, & x \in \left[-2 - \dfrac{\sqrt{5}\varepsilon}{5}, 2 - \dfrac{\sqrt{5}\varepsilon}{5}\right), \\
-4 - \sqrt{\varepsilon^2 - (x-2)^2} & x \in \left[2 - \dfrac{\sqrt{5}\varepsilon}{5}, 2 + \dfrac{2\sqrt{5}\varepsilon}{5}\right), \\
2x - 8 - \sqrt{5}\varepsilon & x \in \left[2 + \dfrac{2\sqrt{5}\varepsilon}{5}, +\infty\right).
\end{cases}
$$

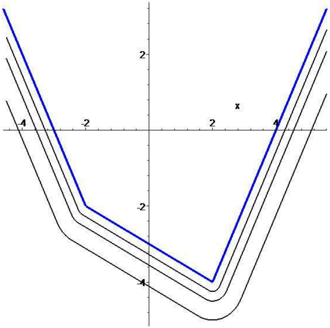

Fig.3. The family functions $f_{\varepsilon}(x)$.

The function f_ε is continuously differentiable on $\mathbb{R}$ and

$$f'_\varepsilon(x) = \begin{cases} -2, & x \in \left(-\infty, -2 - \dfrac{2\sqrt{5}\varepsilon}{5}\right), \\ \dfrac{x+2}{\sqrt{\varepsilon^2 - (x+2)^2}}, & x \in \left[-2 - \dfrac{2\sqrt{5}\varepsilon}{5}, -2 - \dfrac{\sqrt{5}\varepsilon}{5}\right), \\ -\dfrac{1}{2}, & x \in \left[-2 - \dfrac{\sqrt{5}\varepsilon}{5}, 2 - \dfrac{\sqrt{5}\varepsilon}{5}\right), \\ \dfrac{x-2}{\sqrt{\varepsilon^2 - (x-2)^2}} & x \in \left[2 - \dfrac{\sqrt{5}\varepsilon}{5}, 2 + \dfrac{2\sqrt{5}\varepsilon}{5}\right), \\ 2 & x \in \left[2 + \dfrac{2\sqrt{5}\varepsilon}{5}, +\infty\right). \end{cases}$$

Hence $f'_\varepsilon(2) = 0$ and $f_\varepsilon(2) = -4 - \varepsilon$. We have

$$f^*(v) = \begin{cases} -2v + 2, & v \in \left[-2, -\frac{1}{2}\right), \\ 2v + 4, & v \in \left[-\frac{1}{2}, 2\right], \\ +\infty, & \text{at other points.} \end{cases}$$

2. Consider the case when the set D is the segment $[-3, 0]$. Then consider the function

$$\tilde{f}(x) = \max\left\{-2x - 6, -\frac{1}{2}x - 3\right\}, \quad x \in [-3, 1] \subset \mathbb{R}.$$

$$\tilde{f}(x) = \begin{cases} +\infty, & x \in (-\infty, -2), \\ -2x - 6, & x \in \left[-2, -\frac{1}{2}\right), \\ -\frac{1}{2}x - 3, & x \in \left[-\frac{1}{2}, 0\right], \\ +\infty, & x \in \left(-\frac{1}{2}, +\infty\right). \end{cases}$$

Then $D_\varepsilon = [-3 - \varepsilon, \varepsilon]$ (see Fig.4.) and

$$\tilde{f}_\varepsilon(x) = \begin{cases} +\infty, & x \in (-\infty, -3-\varepsilon), \\ -\sqrt{\varepsilon^2 - (x+3)^2}, & x \in \left[-3-\varepsilon, -3-\dfrac{2\sqrt{5}\varepsilon}{5}\right), \\ -2x - 6 - \sqrt{5}\varepsilon & x \in \left[-3-\dfrac{2\sqrt{5}\varepsilon}{5}, -2-\dfrac{2\sqrt{5}\varepsilon}{5}\right), \\ -2 - \sqrt{\varepsilon^2 - (x+2)^2}, & x \in \left[-2-\dfrac{2\sqrt{5}\varepsilon}{5}, -2-\dfrac{\sqrt{5}\varepsilon}{5}\right), \\ -\dfrac{1}{2}x - 3 - \dfrac{\sqrt{5}\varepsilon}{2}, & x \in \left[-2-\dfrac{\sqrt{5}\varepsilon}{5}, -\dfrac{\sqrt{5}\varepsilon}{4}\right), \\ -3 - \sqrt{\varepsilon^2 - x^2}, & x \in \left[-\dfrac{\sqrt{5}\varepsilon}{4}, \varepsilon\right], \\ +\infty, & x \in (\varepsilon, +\infty). \end{cases}$$

The function $\tilde{f}_\varepsilon$ is differentiable for all $x \in (-3-\varepsilon, \varepsilon)$ and $\tilde{f}'_\varepsilon(0) = 0$. As

$$\tilde{f}_\varepsilon(0) = -3-\varepsilon, \quad \tilde{f}_\varepsilon(-3-\varepsilon) = 0, \quad f_\varepsilon(\varepsilon) = -3,$$

then

$$\min_{x \in D_\varepsilon} f_\varepsilon(x) = -3-\varepsilon.$$

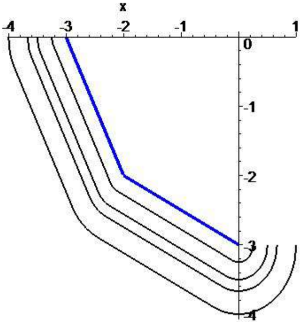

Fig.4. The family $\tilde{f}_{\varepsilon}(x)$.

Theorem 4.

For the function f_ε the following statements

$$\text{dom } f_\varepsilon = \text{dom } f_1 + B_\varepsilon(0_n), \quad \text{epi } f_\varepsilon = \text{epi } f_1 + B_\varepsilon(0_{n+1}),$$

hold.

Theorem 5.

For any fixed $\varepsilon > 0$ the function f_ε is continuously differentiable at each interior point of D_ε.

Corollary 1.

The set epi f_ε is smooth for any positive ε

Theorem 6.

1. For any fixed point x_0 there exists a unique point $z_0 \in D$ for which
$$\varphi_\varepsilon(x_0) = f(z_0) + t_\varepsilon(x_0, z_0).$$
2. $H_\varepsilon = \text{epi } \varphi_\varepsilon = \bigcup_{z \in D} \text{epi } \varphi_\varepsilon(\cdot, z) = \bigcup_{z \in D} H_\varepsilon(z).$
3. $H_\varepsilon = Z_\varepsilon.$

Corollary 2.

For any fixed $\varepsilon > 0$

$$f_\varepsilon(x) = \varphi_\varepsilon(x).$$

Theorem 7.

Let a point $x_0 \in \text{ int} D_\varepsilon$. Then there exists a unique point $z_0 \in D$ for which

$$f'_\varepsilon(x_0) \in \partial f(z_0).$$

Note some properties of functions conjugate to the functions f and f_ε. Let f be a closed proper convex function on $\mathbb{R}^n$. A set

$$\mathrm{dom}\ \partial f = \{x \in \mathbb{R}^n \mid \partial f(x) \neq \emptyset\}$$

and

$$\mathrm{range}\ \partial f = \bigcup_{x \in \mathbb{R}^n} \partial f(x)$$

are called respectively the effective set and the image of ∂f. It is known, that

$$\mathrm{ri}(\mathrm{dom}\ \ f^*) \subset \mathrm{range}\ \partial f \subset \mathrm{dom}\ f^*.$$

Take

$$v \in \text{range } \partial f_\varepsilon \subset \text{dom} f_\varepsilon^*.$$

Then there exists a point $x \in \text{dom} f_\varepsilon$ for which $v \in \partial f_\varepsilon(x)$, therefore,

$$f_\varepsilon(x) + f_\varepsilon^*(v) = \langle x, v \rangle.$$

Consider the point $\bar{x} = [x, f_\varepsilon(x)]$. Find

$$\bar{z} = \arg \min_{\tilde{z} \in X_\varepsilon} ||\tilde{z} - \bar{x}|| = [z, f(z)],$$

Therefore

$$f_\varepsilon'(x) \in \partial f(z), \quad \bar{x} = \bar{z} + \varepsilon\mu(x)[f_\varepsilon'(x), -1], \; x = z + \varepsilon\mu(v) f_\varepsilon'(x).$$

where

$$\mu(x) = \frac{1}{\sqrt{1 + ||f_\varepsilon'(x)||^2}}.$$

Theorem 8.

If the set D is compact and convex, then

$$\min_{x \in D} f(x) = \min_{x \in D_\varepsilon} f_\varepsilon(x) + \varepsilon.$$

Let M be the set of minimizers of f on D, and M_ε be the set of minimizers of f_ε on D_ε.
The case when these sets are empty is not excluded.

Theorem 9.

1. The next equality $M = M_\varepsilon$ holds.
2. If M is not empty set, then

$$f_\varepsilon(z^*) = f(z^*) - \varepsilon \quad \forall z^* \in M.$$

Difference of convex functions

Let $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ be finite convex functions on $\mathbb{R}^n$ and

$$f(x) = f_1(x) - f_2(x).$$

The function f is a quasidifferentiable function.

Quasidifferentiable functions

Let a function f be defined on $\mathbb{R}^n$ and be directionally differentiable at a point $x \in \mathbb{R}^n$ and its directional derivative $f'(x, g)$ can be represented in the form

$$f'(x,g) = \lim_{\lambda \downarrow 0} \frac{f(x+\lambda g) - f(x)}{\lambda} = \max_{v \in \underline{\partial} f(x)} \langle v, g \rangle + \min_{w \in \overline{\partial} f(x)} \langle w, g \rangle.$$

Here $\underline{\partial} f(x) \subset \mathbb{R}^n$, $\overline{\partial} f(x) \subset \mathbb{R}^n$ are convex compact sets in $\mathbb{R}^n$.

The function f is called a quasidifferentiable at a point $x \in \mathbb{R}^n$. A pair of sets $\mathcal{D}f(x) = [\underline{\partial} f(x), \overline{\partial} f(x)]$ is called a quasidifferential of a quasidifferentiable function f at x.
The set $\underline{\partial} f(x) \subset \mathbb{R}^n$ is called a subdifferential of f at x, the set $\overline{\partial} f(x) \subset \mathbb{R}^n$ is called a superdifferential of f at x.
Differentiable, convex, concave functions, maximum functions are quasidifferentiable functions.
As the function f is quasidifferentiable on $\mathbb{R}^n$ and

$$\mathcal{D}f(x) = [\partial f_1(x), -\partial f_2(x)]$$

is its quasidifferential at a point $x \in \mathbb{R}^n$, where $\partial f_i(x)$ are the subdifferentials of convex functions $f_i(x)$, $i = 1, 2$, at the point $x \in \mathbb{R}^n$ in the sense of Convex Analysis.

Let's consider the optimization problem: find

$$\inf_{x \in \mathbb{R}^n} f(x).$$

The following necessary optimality conditions for the function f on $\mathbb{R}^n$ hold.

Theorem 10.

For a point $x^* \in \mathbb{R}^n$ to be a minimizer of the function f on $\mathbb{R}^n$, it is necessary, that

$$\partial f_2(x^*) \subset \partial f_1(x^*). \tag{2}$$

For a point $x^* \in \mathbb{R}^n$ to be a maximizer of the function f on $\mathbb{R}^n$, it is necessary, that

$$\partial f_1(x^*) \subset \partial f_2(x^*). \tag{3}$$

If the inclusion

$$\partial f_2(x^*) \subset \text{int}\ \partial f_1(x^*)$$

holds at the point $x^* \in \mathbb{R}^n$ then this point is a strict local minimizer of the function f on $\mathbb{R}^n$.

If the inclusion

$$\partial f_1(x^*) \subset \text{int}\ \partial f_2(x^*)$$

is satisfied at the point $x^* \in \mathbb{R}^n$ then this point is a strict local maximizer of the function f *on* $\mathbb{R}^n$.

A point x^* is called an *inf*− stationary point of f if inclusion (2) holds. A point x^{**} is called a *sup*− stationary point of f if inclusion (3) holds.
We say that a point x^* is Clark's stationary point of function f on $\mathbb{R}^n$, if the next condition

$$\partial f_1(x^*) \bigcap \partial f_2(x^*) \neq \emptyset$$

holds. It is obvious that inf − and sup-stationary points of the function f on $\mathbb{R}^n$ are also Clark's stationary points of f on $\mathbb{R}^n$.

Lemma 2.

Fix any point $x \in \mathbb{R}^n$, then

$$f_1(x) - f_2(x) \leq f_2^*(v) - f_1^*(v) \qquad \forall v \in \partial f_1(x), \tag{4}$$

$$f_1(x) - f_2(x) \geq f_2^*(v) - f_1^*(v) \qquad \forall v \in \partial f_2(x), \tag{5}$$

$$f_1(x) - f_2(x) = f_2^*(v) - f_1^*(v) \qquad \forall v \in \partial f_1(x) \cap \partial f_2(x). \tag{6}$$

Denote by

$$f^o(v) = f_2^*(v) - f_1^*(v), \quad v \in \mathbb{R}^n.$$

If a point $v \notin \text{dom } f_1^* \cup \text{dom } f_2^*$, then we face with the case $+\infty - \infty$.

Therefore in different cases under considering of certain extremal properties, we will define this function on different depending on the situation

1. If the point $x^* \in \mathbb{R}^n$ is Clark's stationary point of the function f on $\mathbb{R}^n$, then the equality

$$f(x^*) = f^o(v) \quad \forall v \in \partial f_1(x^*) \cap \partial f_2(x^*)$$

holds.

2. If the point $x^* \in \mathbb{R}^n$ is an inf-stationary point of the function f on $\mathbb{R}^n$, then the equality

$$f(x^*) = f^o(v^*) \quad \forall v^* \in \partial f_2(x^*) \tag{7}$$

holds.

3. If the point $x^* \in \mathbb{R}^n$ is sup-stationary point of the function f on $\mathbb{R}^n$, then the equality

$$f(x^*) = f^o(v^*) \quad \forall v^* \in \partial f_1(x^*) \tag{8}$$

holds.

Example 4.

Consider a function

$$f(x) = f_1(x) - f_2(x) = |x_1| - |x_2|, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

However, it is Clark's stationary point of the function f on $\mathbb{R}^2$. Define conjugate functions of f_1 and f_2. We have

$$f_1^*(v) = \begin{cases} 0, & v \in \text{co } \{(1,0),(-1,0)\}, \\ +\infty, & v \notin \text{co } \{(1,0),(-1,0)\}, \end{cases}$$

$$f_2^*(v) = \begin{cases} 0, & v \in \text{co } \{(0,1),(0,-1)\}, \\ +\infty, & v \notin \text{co } \{(0,1),(0,-1)\}, \end{cases}$$

As $\partial f_1(x^*) = \text{co } \{(1,0),(-1,0)\}, \quad \partial f_2(x^*) = \text{co } \{(0,1),(0,-1)\}$, then

$0_2 \in \partial f_1(x^*) \cap \partial f_2(x^*) = \text{co } \{(1,0),(-1,0)\} \cap \text{co } \{(0,1),(0,-1)\}$.

Therefore $f(x^*) = 0 = f^o(0_2)$.
From conditions (6) it follows that if the point x^* is Clarke's stationary point of the function f on $\mathbb{R}^n$, then the next relation

$$\partial f_1^*(v^*) \cap \partial f_2^*(v^*) \neq \emptyset \quad \forall v^* \in \partial f_1(x^*) \cap \partial f_2(x^*)$$

is valid

Example 5.

Consider the function

$$f_1(x) = \begin{cases} x^3, & x \geq 0, \\ 0, & x < 0, \end{cases} \quad , \quad f_2(x) = \begin{cases} 0, & x \geq 0, \\ -x^3, & x < 0, \end{cases} \qquad x \in \mathbb{R}.$$

Note that the functions f_1 and f_2 are convex and continuously differentiable. Then $f(x) = x^3$. The function f has a unique stationary point $x^* = 0$ and

$$f_1'(x^*) = 0, \quad f_2'(x^*) = 0.$$

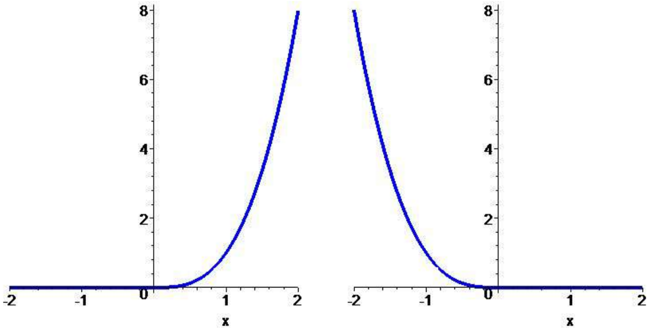

Fig.5. The functions f_1 and f_2.

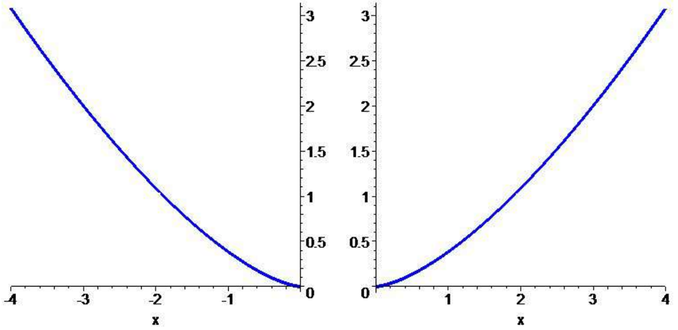

Fig.5. The functions f_2^* and f_1^*.

Calculate f_1^* and f_2^*. We have

$$f_1^*(v) = \begin{cases} \dfrac{2v\sqrt{v}}{3\sqrt{3}}, & v \geq 0, \\ +\infty, & v < 0, \end{cases}, \quad f_2^*(v) = \begin{cases} +\infty, & v > 0, \\ -\dfrac{2v\sqrt{|v|}}{3\sqrt{3}}, & v \leq 0, \end{cases} \quad v \in \mathbb{R}.$$

Therefore,

$$f^o(v) = \begin{cases} +\infty, & v > 0, \\ 0, & v = 0, \\ -\infty, & v < 0. \end{cases}$$

The function f^o is finite only in a single point $v^* = 0$. Find subdifferentials of functions f_1^* and f_2^* at the point v^*

$$\partial f_1^*(0) = (-\infty, 0] \subset \mathbb{R}, \quad \partial f_2^*(0) = [0, +\infty) \subset \mathbb{R}.$$

It is obvious that $\partial f_1^*(0) \cap \partial f_2^*(0) = 0$.
Note the fact that if we calculate the function conjugate to the function f, then $f^*(v) = +\infty \quad \forall v \in \mathbb{R}$.

From (4) and (5) it follows:
1) if dom $f_2^* \not\subset$ dom f_1^*, the function f are unbounded from below,
2) if dom $f_1^* \not\subset$ dom f_2^*, then the function f unbounded from above.

Note that in the points not belonging to the set dom f_1^*, we face with the case $+\infty - \infty$, therefore, under minimizing the function f on $\mathbb{R}^n$, we define the function f^o on the complement of the set dom f_1^* to the whole space by the value $+\infty$.
Namely, put

$$f_-^o(v) = \begin{cases} f^o(v), & v \in \text{dom } f_1^*, \\ +\infty, & v \notin \text{dom } f_1^*. \end{cases} .$$

Theorem 11.

Let a point $x^* \in \mathbb{R}^n$ *be a global minimizer of the function* f on $\mathbb{R}^n$, then any subgradient $v^* \in \partial f_2(x^*)$ is a global minimizer of the function f_-^o on $\mathbb{R}^n$.

Theorem 12.

Let a point $x^* \in \mathbb{R}^n$ be Clark's stationary point of the function f *on* $\mathbb{R}^n$. Then, if there is a global minimizer

$$v^* \in \partial f_1(x^*) \cap \partial f_2(x^*),$$

of the function f_-^o on $\mathbb{R}^n$, then the point x^* is a global minimizer of the function f *on* $\mathbb{R}^n$.

Corollary 3.

If there is a global minimizer $x^* \in \mathbb{R}^n$ of the function f on $\mathbb{R}^n$ then there is a point $v^* \in \mathbb{R}^n$, which is a global minimizer of the function f_-^o on $\mathbb{R}^n$. In this case the relation

$$\min_{x \in \mathbb{R}^n} f(x) = \min_{v \in \mathbb{R}^n} f_-^o(v)$$

is valid.
If the function f achieves at some point $x^* \in \mathbb{R}^n$ its global minimum on $\mathbb{R}^n$, then dom $f_2^* \subset$ dom f_1^*.

Smooth approximation of d.c. functions

Let f_1, f_2 be convex functions on $\mathbb{R}^n$ and

$$f(x) = f_1(x) - f_2(x), \quad x \in \mathbb{R}^n.$$

Fix $\varepsilon > 0$ and form functions

$$f_\varepsilon(x) = f_{1\varepsilon}(x) - f_{2\varepsilon}(x),$$

where

$$f_{1\varepsilon}(x) = (f_1 \Box t_\varepsilon)(x), \quad f_{2\varepsilon}(x) = (f_2 \Box t_\varepsilon)(x),$$

$$t_\varepsilon(x) = \begin{cases} -\sqrt{\varepsilon^2 - ||x||^2}, & ||x|| \le \varepsilon, \\ +\infty, & ||x|| > \varepsilon, \end{cases} \qquad x \in \mathbb{R}^n.$$

The function f_ε is continuous differentiable on $\mathbb{R}^n$.

Theorem 13.

If a point $x^* \in \mathbb{R}^n$ is a stationary point of the function f_ε on $\mathbb{R}^n$ $(f'_{1\varepsilon}(x^*) = f'_{2\varepsilon}(x^*))$, then at the point

$$z^* = x^* - \frac{\varepsilon}{\sqrt{1 + ||f'_{1\varepsilon}(x^*)||^2}} f'_{1\varepsilon}(x^*)$$

the next intersection

$$\partial f_1(x^*) \cap \partial f_2(x^*) \neq \emptyset$$

holds and

$$f(z^*) = f_\varepsilon(x^*), \quad f'_{1\varepsilon}(x^*) \in \partial f_1(z^*) \cap \partial f_2(z^*).$$

Theorem 14.

Let a point x^* be a global minimizer of f_ε on $\mathbb{R}^n$, then the point

$$z^* = x^* - \frac{\varepsilon}{\sqrt{1 + ||f'_{1\varepsilon}(x^*)||^2}} f'_{1\varepsilon}(x^*)$$

is a global minimizer of f on $\mathbb{R}^n$ and

$$f(z^*) = f_\varepsilon(x^*),$$

$$f'_{1\varepsilon}(x^*) \in \partial f_2(z^*) \subset \partial f_1(z^*).$$

Theorem 15.

Let a point z^* be a global minimizer of the function f on $\mathbb{R}^n$. Then the point

$$x_v^* = z^* + \frac{\varepsilon}{\sqrt{1 + ||v||^2}} v$$

is also a global minimizer of the function f_ε on $\mathbb{R}^n$ for each $v \in \partial f_2(z^*)$ and

$$f(z^*) = f_\varepsilon(x_v^*),$$

$$v = f'_{1\varepsilon}(x_v^*) = f'_{2\varepsilon}(x_v^*) \quad \forall v \in \partial f_2(z^*) \subset \partial f_1(z^*).$$

Thank you for your attention