Every compact convex subset of matrices is the Clarke Jacobian of some Lipschitzian mapping

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Outline of the talk

- Introduction: basic concepts, the question and our answer
- Five lemmas: ray-fish, ray-fish colony, ray-fish colony for a line segment,
  ray-fish colony for a polygonal chain, and corona
- Finale: the main result and its proof

Introduction

Introduction: Basic concepts

Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitzian mapping; that is, a mapping such that

$$\|f(x) - f(y)\| \leq L\|x - y\| \qquad \text{for every} \quad x, y \in \mathbb{R}^n$$

for some constant $L > 0$, where the norms $\|\cdot\|$ are Euclidean.

The Euclidean vector space $\mathbb{R}^n$ and $\mathbb{R}^m$ of dimension n and m, respectively, is identified with the space $\mathbb{R}^{n\times 1}$ and $\mathbb{R}^{m\times 1}$, respectively, by convention. Therefore, a vector $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$ is understood as column vector of n and m, respectively, real numbers.

Introduction: Basic concepts

Every matrix $A \in \mathbb{R}^{m \times n}$ induces a linear mapping $L_A: \mathbb{R}^n \to \mathbb{R}^m$ by

$$L_A: x \mapsto Ax \qquad \text{for} \quad x \in \mathbb{R}^n$$

and every linear mapping $L: \mathbb{R}^n \to \mathbb{R}^m$ is induced by a matrix $A_L \in \mathbb{R}^{m \times n}$ so that

$$L(x) = A_L x \qquad \text{for} \quad x \in \mathbb{R}^n$$

This is the reason why we shall identify each matrix $A \in \mathbb{R}^{m \times n}$ with the respective linear mapping which it induces, and vice versa.

Introduction: Basic concepts

By **Rademacher's Theorem**, the given Lipschitzian mapping $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is Gâteaux (therefore: Fréchet) differentiable almost everywhere with respect to the n-dimensional Lebesgue measure on $\mathbb{R}^n$.

That is, the Jacobian matrix

$$\mathrm{J}f(x) = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_m(x)}{\partial x_1} & \frac{\partial f_m(x)}{\partial x_2} & \cdots & \frac{\partial f_m(x)}{\partial x_n} \end{pmatrix}$$

is defined for almost all $x \in \mathbb{R}^n$.

The Jacobian matrix $\mathrm{J}f(x)$ is identified with the Gâteaux derivative, which is the linear mapping $f'(x)\colon \mathbb{R}^n \to \mathbb{R}^m$.

Introduction: Basic concepts

The **Bouligand Jacobian** of a Lipschitzian mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point $x_0 \in \mathbb{R}^n$ is the set

$$\partial_{\mathrm{B}} f(x_0) = \Big\{ M \in \mathbb{R}^{m \times n} : \exists (x_k)_{k=1}^{\infty} \subset \mathbb{R}^n \colon \lim_{k\to\infty} x_k = x_0 \,, \text{ function } f \text{ is differentiable at each } x_k \text{ and } \lim_{k\to\infty} \mathrm{J}f(x_k) = M \Big\}$$

Since the set $\partial_{\mathrm{B}} f(x_0)$ is a collection of matrices, it is identified with the collection of the corresponding linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Introduction: Basic concepts

The **Clarke Jacobian** of a Lipschitzian mapping $f\colon \mathbb{R}^n \to \mathbb{R}^m$

at a point $x_0 \in \mathbb{R}^n$ is the set

$$\partial f(x_0) = \mathrm{co}\, \partial_{\mathrm{B}} f(x_0)$$

It is easy to see that the Clarke Jacobian $\partial f(x_0)$ is:

- non-empty
- compact (that is closed & bounded)
- convex

The Clarke Jacobian $\partial f(x_0)$ is also seen as

a collection of linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Introduction: The Question

Given a non-empty compact convex set $\mathcal{P} \subset \mathbb{R}^{m \times n}$ of matrices,

is this set the Clarke Jacobian of some Lipschitzian mapping $g: \mathbb{R}^n \to \mathbb{R}^m$

at some point $x_0 \in \mathbb{R}^n$?

In other words, characterize those non-empty compact convex sets $\mathcal{P} \subset \mathbb{R}^{m \times n}$

of matrices that are Clarke Jacobians of some Lipschitzian mappings.

Introduction: Our Answer

Given a non-empty compact convex set $\mathcal{P} \subset \mathbb{R}^{m \times n}$ of matrices,

there exists a Lipschitzian mapping $g: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\partial g(0) = \mathcal{P}$$

We actually prove more…

Introduction: Our Answer II

Consider a linear subspace $\{0\} \subsetneq W \subset \mathbb{R}^n$

In the following, we identify every matrix $M \in \mathbb{R}^{m \times n}$ and the linear mapping $L_M: \mathbb{R}^n \to \mathbb{R}^m$, defined by $L_M: x \mapsto Mx$ for $x \in \mathbb{R}^n$, which it induces; that is, we use the same symbol "M" for both the matrix M and the mapping L_M.

By $M_{|W}$ we denote the linear mapping $M = L_M$ restricted onto the subspace W, that is the mapping

$$M_{|W}: W \longrightarrow \mathbb{R}^m$$

$$M_{|W}: x \longmapsto Mx \qquad \text{for} \quad x \in W$$

Introduction: Our Answer II

Consider the linear subspace $\{0\} \subsetneq W \subset \mathbb{R}^n$.

Consider also the given non-empty compact convex set $\mathcal{P} \subset \mathbb{R}^{m \times n}$ of matrices.

By $\mathcal{P}_{|W}$ we denote the collection of the restricted linear mappings

$$\mathcal{P}_{|W} = \left\{ M_{|W} : M \in \mathcal{P} \right\}$$

where

$$M_{|W} : W \longrightarrow \mathbb{R}^m$$

$$M_{|W} : x \longmapsto Mx \qquad \text{for} \quad x \in W \qquad \text{for every} \quad M \in \mathcal{P}$$

Introduction: Our Answer II

Consider the linear subspace $\{0\} \subsetneq W \subset \mathbb{R}^n$.

Consider also a Lipschitzian mapping $g: \mathbb{R}^n \to \mathbb{R}^m$.

By $g_{|W}$ we denote the restriction of g onto W, that is the mapping

$$g_{|W}: W \longrightarrow \mathbb{R}^m$$

$$g_{|W}: x \longmapsto g(x)$$

Recall that we identify the Clarke Jacobian $\partial g(0) \subset \mathbb{R}^{m \times n}$ with the respective collection of the linear mappings $M: \mathbb{R}^n \to \mathbb{R}^m$ for $M \in \partial g(0)$.

Therefore, we can define the Clarke Jacobian $\partial g_{|W}(0)$ accordingly.

Introduction: Our Answer II

Let $\mathcal{P} \subset \mathbb{R}^{m \times n}$ be a non-empty compact convex set of matrices.

Then there exists a Lipschitzian mapping $g \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m$, with $g(0) = 0$, such that, for every linear subspace $\{0\} \subsetneq W \subset \mathbb{R}^n$, the Clarke Jacobian

$$\partial g_{|W}(0) = \mathcal{P}_{|W}$$

Five Lemmas

The Ray-Fish Construction: Introduction I

We define that a mapping $f: \mathbb{R}^n \to \mathbb{R}^m$ is **finitely piecewise affine** if there are finitely many pairwise disjoint non-empty open sets $\Omega_1, \dots, \Omega_k \subset \mathbb{R}^n$ such that $\mathbb{R}^n \setminus \bigcup_{i=1}^k \Omega_i$ is Lebesgue negligible and there are matrices $M_1, \dots, M_k \in \mathbb{R}^{m \times n}$ and constant vectors $c_1, \dots, c_k \in \mathbb{R}^m$ such that

$$f(x) = \begin{cases} M_1 x + c_1, & \text{if } x \in \Omega_1 \\ M_2 x + c_2, & \text{if } x \in \Omega_2 \\ \quad \dots & \quad \dots \\ M_k x + c_k, & \text{if } x \in \Omega_k \end{cases}$$

Observe that this mapping f is also Lipschitzian with the Lipschitz constant $\max\{\|M_1\|, \|M_2\|, \dots, \|M_k\|\}$ and that its derivative $f'(x) = M_i$ for $x \in \Omega_i$ for $i = 1, 2, \dots, k$.

The Ray-Fish Construction: Introduction II

Recall that the **rank** of a matrix $A \in \mathbb{R}^{m \times n}$ is

the maximum number of the rows of A that are linearly independent.

Now, consider two matrices $P, Q \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(P - Q) = 1.$

Observe then that the set

$$H_{PQ} = \{\, x \in \mathbb{R}^n : Px = Qx \,\}$$

is a hyperplane.

We call it **the hyperplane of the continuous contact** of the matrices P and Q.

The Ray-Fish Construction: An Exercise

Consider two matrices $P, Q \in \mathbb{R}^{m \times n}$ with $\operatorname{rank}(P - Q) = 1$.

Then there exists a row vector $u^{\mathrm{T}} \in \mathbb{R}^{1 \times n}$ such that

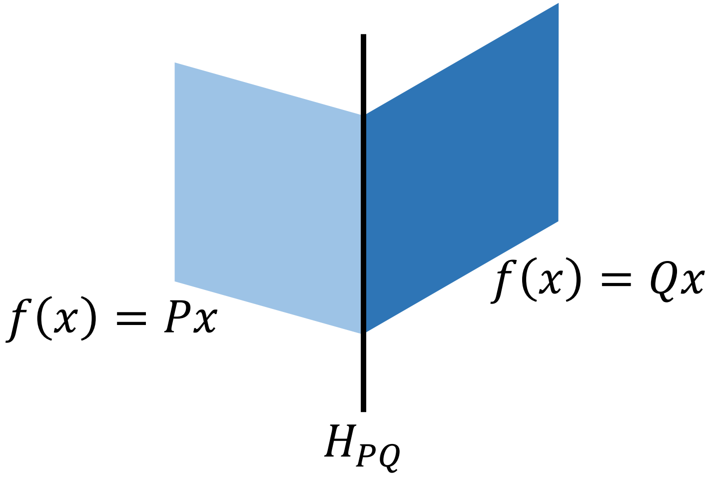
$$H_{PQ} = \{ x \in \mathbb{R}^n : u^{\mathrm{T}} x = 0 \}$$

Define the mapping $f\colon \mathbb{R}^n \to \mathbb{R}^m$ by

$$f(x) = \begin{cases} Px, & \text{if } u^{\mathrm{T}} x \leq 0 \\ Qx, & \text{if } u^{\mathrm{T}} x \geq 0 \end{cases}$$

and observe that f is Lipschitzian

and piecewise linear.



The Ray-Fish Construction: Ray-Fish Lemma 1

Let $\alpha > 0$ and let $P, Q \in \mathbb{R}^{m \times n}$ be two matrices with $\operatorname{rank}(P - Q) \leq 1$.

Then there exists a finitely piecewise affine Lipschitzian mapping $f_{\alpha,PQ} \colon \mathbb{R}^n \to \mathbb{R}^m$

such that

- $f_{\alpha,PQ}(x) = Px$ for all $x \in \mathbb{R}^n \setminus \mathbb{B}_n$, where $\mathbb{B}_n$ denotes the closed unit ball in $\mathbb{R}^n$,
- $f'_{\alpha,PQ}(x) = Q$ for all $x \in \Omega_{\alpha,PQ}$, where $\Omega_{\alpha,PQ} \subset \mathbb{B}_n$ is a non-empty open set,
- and

$$\operatorname{dist}\big(f'_{\alpha,PQ}(x), \{P, Q\}\big) < \alpha$$

whenever $f_{\alpha,PQ}$ is differentiable at $x \in \mathbb{R}^n$.

The Ray-Fish Construction

Let $\alpha > 0$ and let $P, Q \in \mathbb{R}^{m \times n}$ be two matrices with $\mathrm{rank}(P - Q) \leq 1$.

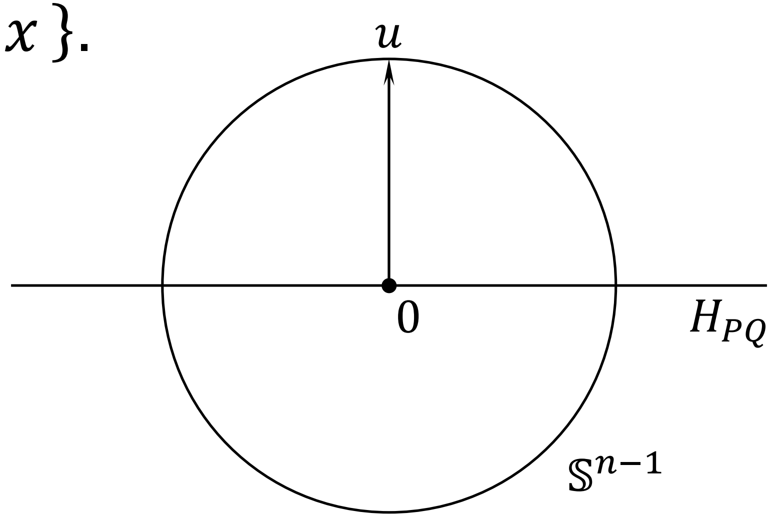
- If $\mathrm{rank}(P - Q) = 0$, then $P = Q$ and the mapping $f_{\alpha,PQ}(x) \coloneqq Px$ works.
- Consider that $\mathrm{rank}(P - Q) = 1$ in the following therefore.

Consider the contact hyperplane $H_{PQ} = \{\, x \in \mathbb{R}^n : Px = Qx \,\}$.

Let $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$.

Pick a vector $u \in \mathbb{S}^{n-1}$ such that $u \perp H_{PQ}$.

Notice that $H_{PQ} = \{\, x \in \mathbb{R}^n : u^{\mathrm{T}} x = 0 \,\}$.



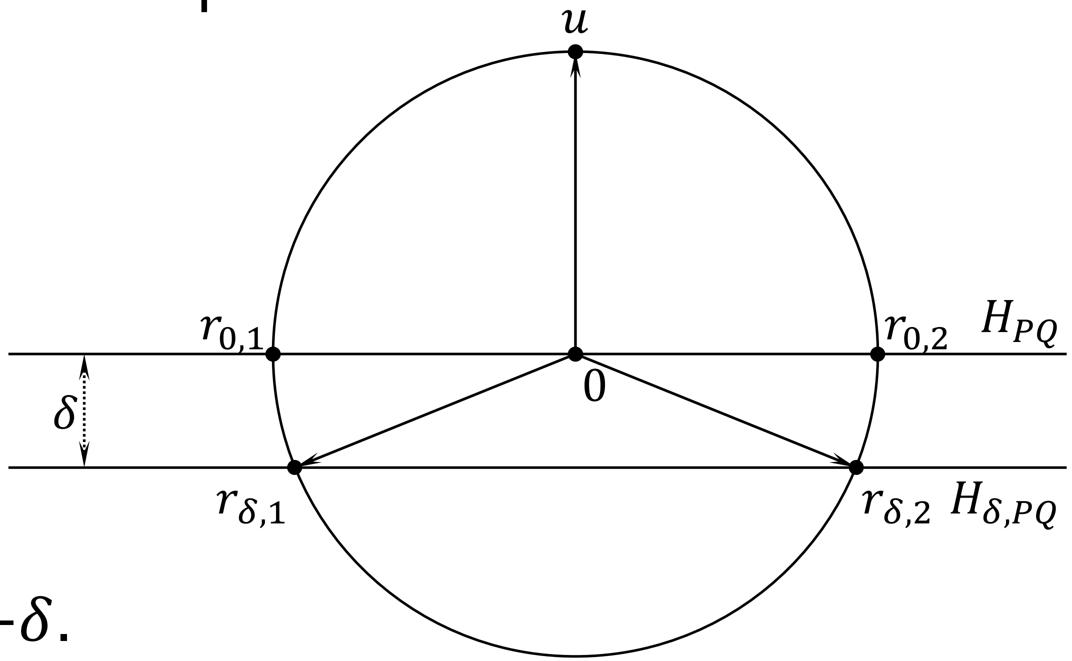
The Ray-Fish Construction

Let $\alpha > 0$ and let $P, Q \in \mathbb{R}^{m \times n}$ be two matrices with $\mathrm{rank}(P - Q) = 1$.

Consider any $\delta \in (0, 1)$ and define $H_{\delta,PQ} := \{\, x \in \mathbb{R}^n : u^{\mathrm{T}} x = -\delta \,\}$.

Choose any points $r_{0,1}, r_{0,2} \dots, r_{0,n} \in H_{PQ} \cap \mathbb{S}^{n-1}$ such that their convex hull

$\mathrm{co}\{r_{0,1}, r_{0,2}, \dots, r_{0,n}\}$ is a regular $(n-1)$-dimensional simplex.



Let $r_{\delta,j} \in H_{\delta,PQ} \cap \mathbb{S}^{n-1}$ be the unique point

such that $\lambda r_{\delta,j} + (1 - \lambda)u = \mu r_{0,j}$ for some

$\lambda, \mu \in (0, 1)$ for $j = 1, 2, \dots, n$.

If $n = 1$, then let $u := 1$, $r_{0,1} := 0$ and $r_{\delta,q} := -\delta$.

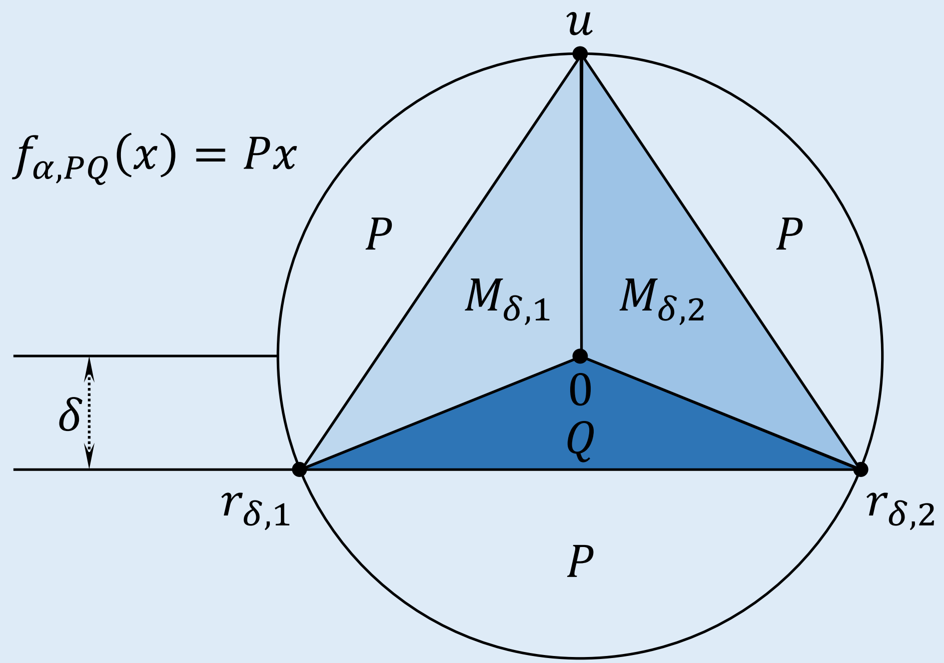
The Ray-Fish Construction

Let $\alpha > 0$ and let $P, Q \in \mathbb{R}^{m\times n}$ be two matrices with $\operatorname{rank}(P - Q) = 1$.

Define

$$f_{\alpha,PQ}(x) = \begin{cases} Px, & \text{if } x \in \mathbb{R}^n \setminus \operatorname{co}\{r_{\delta,1}, \dots, r_{\delta,n}, u\} \supset \mathbb{R}^n \setminus \mathbb{B}_n \\ Qx - \delta(P-Q)u, & \text{if } x \in \operatorname{co}\{r_{\delta,1}, \dots, r_{\delta,n}, 0\} \\ M_{\delta,j}x - \delta(P-Q)u, & \text{if } x \in \operatorname{co}\{r_{\delta,1}, \dots, r_{\delta,j-1}, u, r_{\delta,j+1}, \dots, r_{\delta,n}\} \end{cases}$$

where the matrices $M_{\delta,1}, M_{\delta,2}, \dots, M_{\delta,n}$ are to be found so that the mapping $f_{\alpha,PQ}$ is well-defined, hence continuous and Lipschitzian.



The Ray-Fish Construction

Let $\alpha > 0$ and let $P, Q \in \mathbb{R}^{m\times n}$ be two matrices with $\operatorname{rank}(P - Q) = 1$.

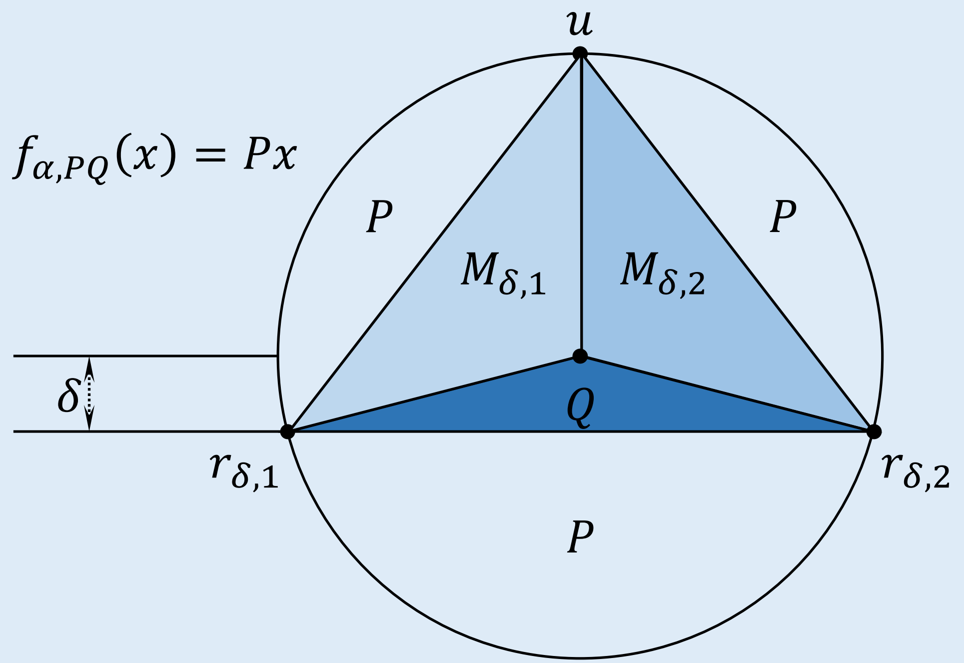
A few elementary calculations show that

$$M_{\delta,j} \longrightarrow P \qquad \text{as} \quad \delta \downarrow 0$$

Find $\delta \in (0,1)$ so small that

$$\|M_{\delta,j} - P\| < \alpha \qquad \text{for} \quad j = 1, \dots, n$$

We are done thus.



Ray-Fish

Source: https://commons.wikimedia.org/wiki/File:Rays_(32199123686).jpg

Ray-Fish

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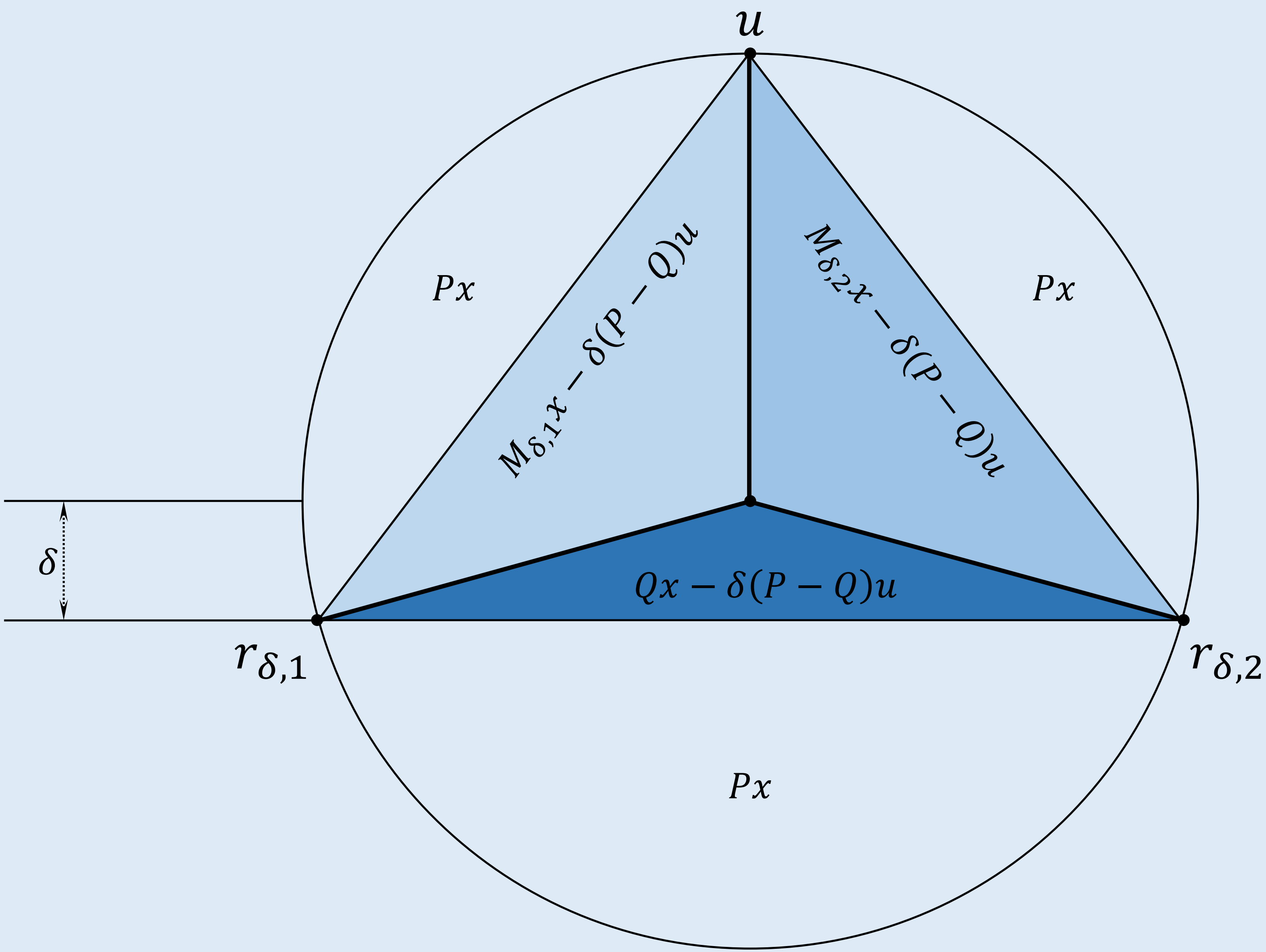
The Ray-Fish



u
Px
Px
$M_{\delta,1}x - \delta(P-Q)u$
$M_{\delta,2}x - \delta(P-Q)u$
δ
$Qx - \delta(P-Q)u$
$r_{\delta,1}$
$r_{\delta,2}$
Px

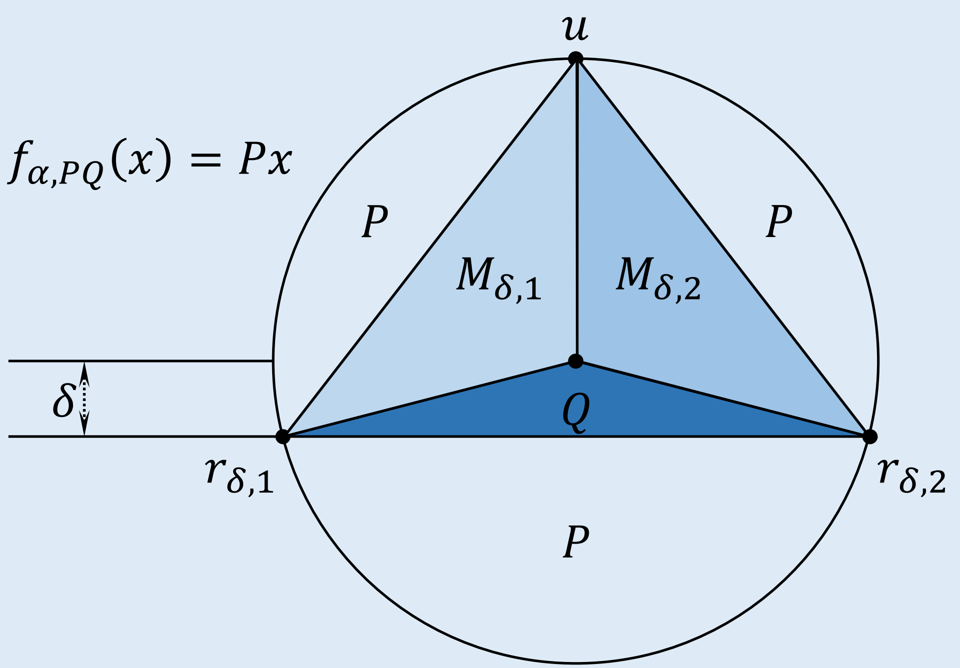
The Ray-Fish Construction

Let $\alpha > 0$ and let $P, Q \in \mathbb{R}^{m\times n}$ be two matrices with $\operatorname{rank}(P - Q) = 1$.

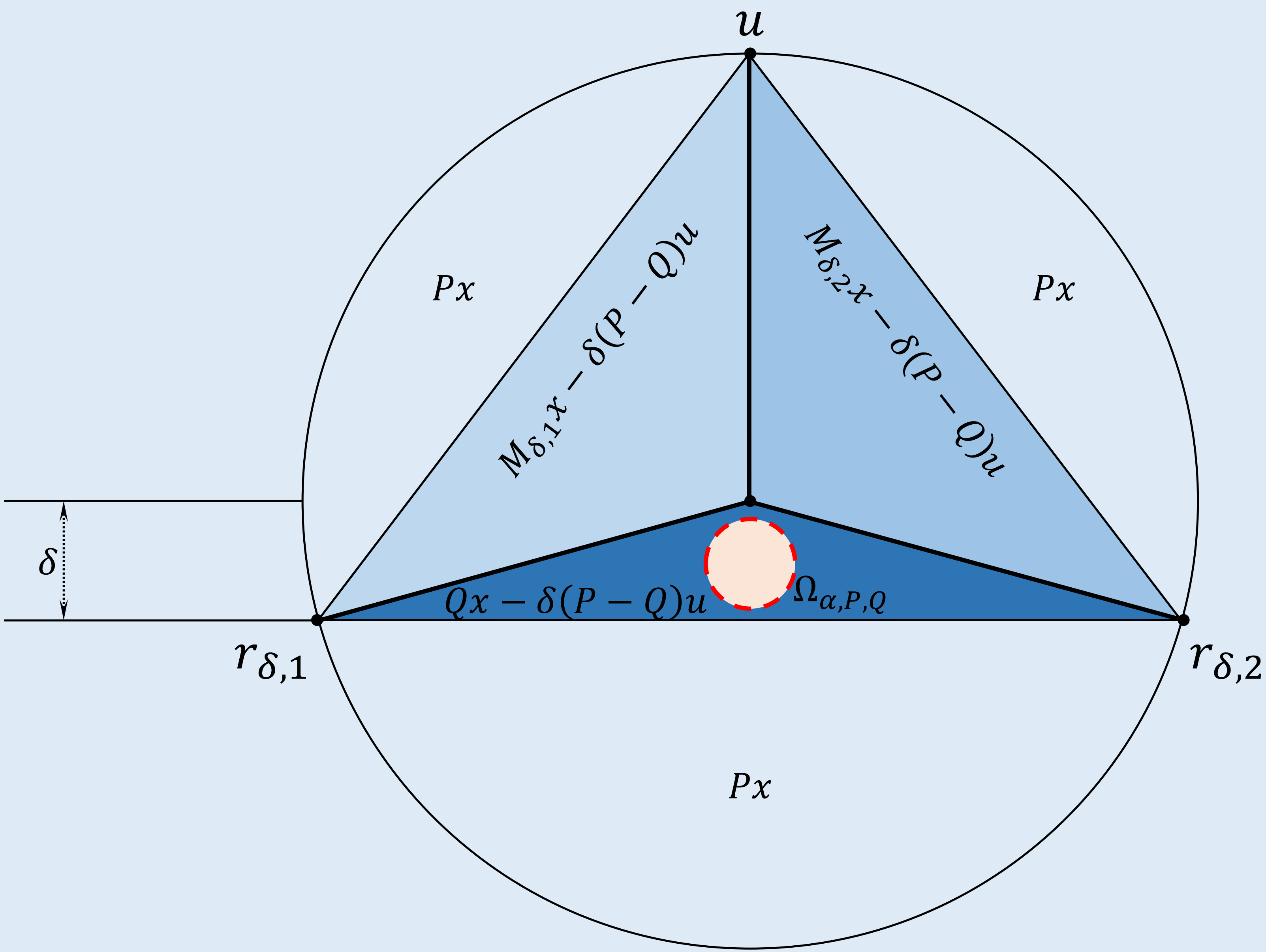
Notice that:

- $f'_{\alpha,PQ}(x) \in \{Q, P, M_{\delta,1}, M_{\delta,2}\}$ whenever $f_{\alpha,PQ}$ is differentiable at $x \in \mathbb{R}^n$
- $\partial f_{\alpha,PQ}(0) = \operatorname{co}\{Q, M_{\delta,1}, M_{\delta,2}\}$
- for any subspace $\{0\} \subsetneq W \subset \mathbb{R}^n$,

$$\partial \left(f_{\alpha,PQ}\right)_{|W}(0) = \\ = \operatorname{co}\left\{Q_{|W}, \left(M_{\delta,1}\right)_{|W}, \left(M_{\delta,2}\right)_{|W}\right\}$$



The Ray-Fish



u
Px
$M_{\delta,1}x - \delta(P-Q)u$
$M_{\delta,2}x - \delta(P-Q)u$
Px
δ
$Qx - \delta(P-Q)u$
$\Omega_{\alpha,P,Q}$
$r_{\delta,1}$
$r_{\delta,2}$
Px

The Recursive Ray-Fish Construction: Ray-Fish Colony Lemma 2

Let $\beta > 0$ and let $Q_0, \dots, Q_k \in \mathbb{R}^{m \times n}$ be with $\operatorname{rank}\left(Q_j - Q_{j+1}\right) \leq 1$ for $j = 0, \dots, k-1$.

Then there exists a Lipschitzian mapping $g_{\beta,Q_0 \dots Q_k}: \mathbb{R}^n \to \mathbb{R}^m$ such that it is finitely piecewise affine and

- $g_{\beta,Q_0 \dots Q_k}(x) = Q_0 x$ for all $x \in \mathbb{R}^n \setminus \mathbb{B}_n$,
- $g'_{\beta,Q_0 \dots Q_k}(x) = Q_k$ for all x from a non-empty open set $\Omega_{\beta,Q_0 \dots Q_k} \subset \mathbb{B}_n$,
- and

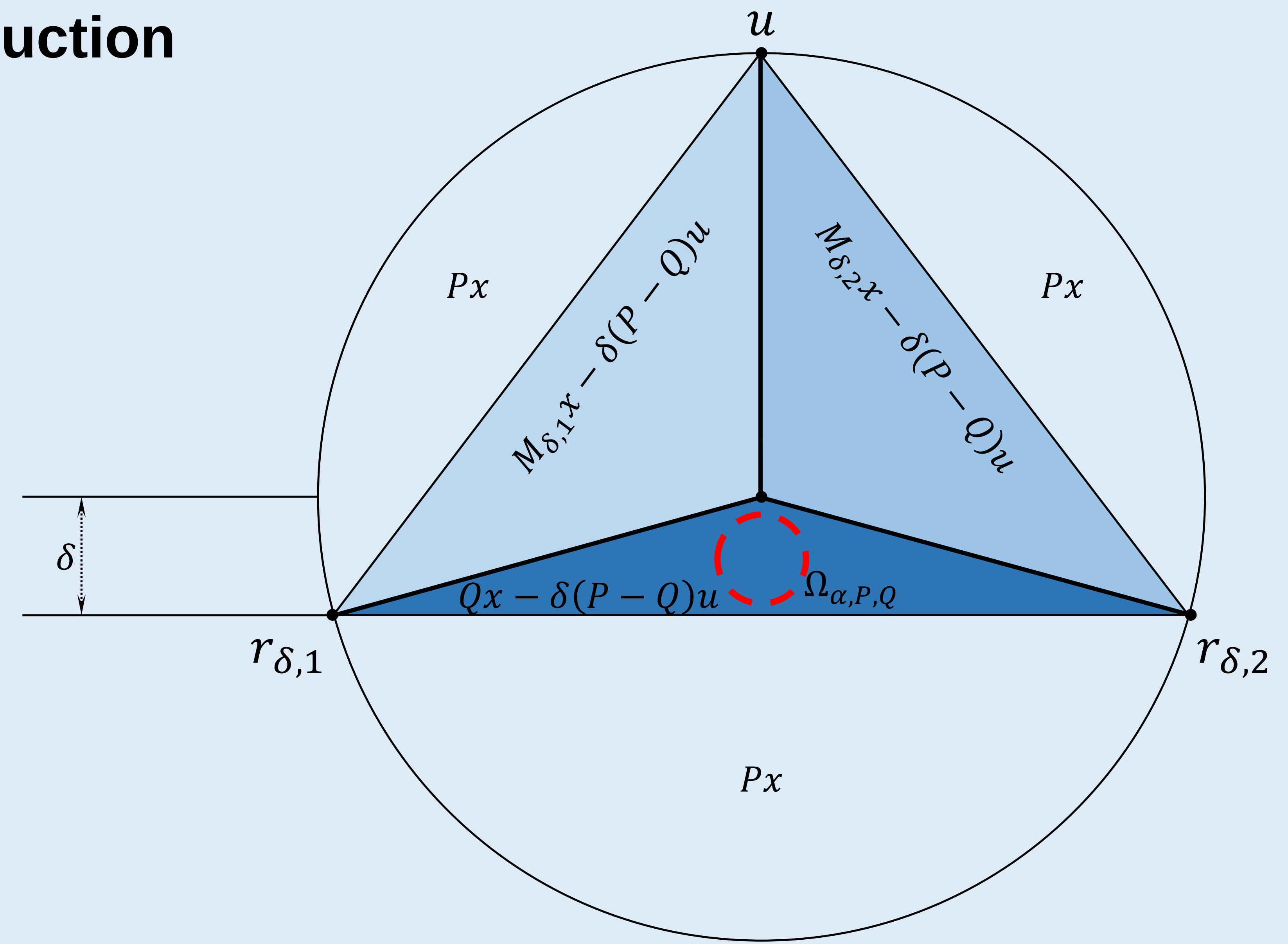
$$\operatorname{dist}\left(g'_{\beta,Q_0 \dots Q_k}(x), \{Q_0, \dots, Q_k\}\right) < \beta$$

whenever $g_{\beta,Q_0 \dots Q_k}$ is differentiable at $x \in \mathbb{R}^n$.

The Recursive Ray-Fish Construction

$P \coloneqq Q_0$

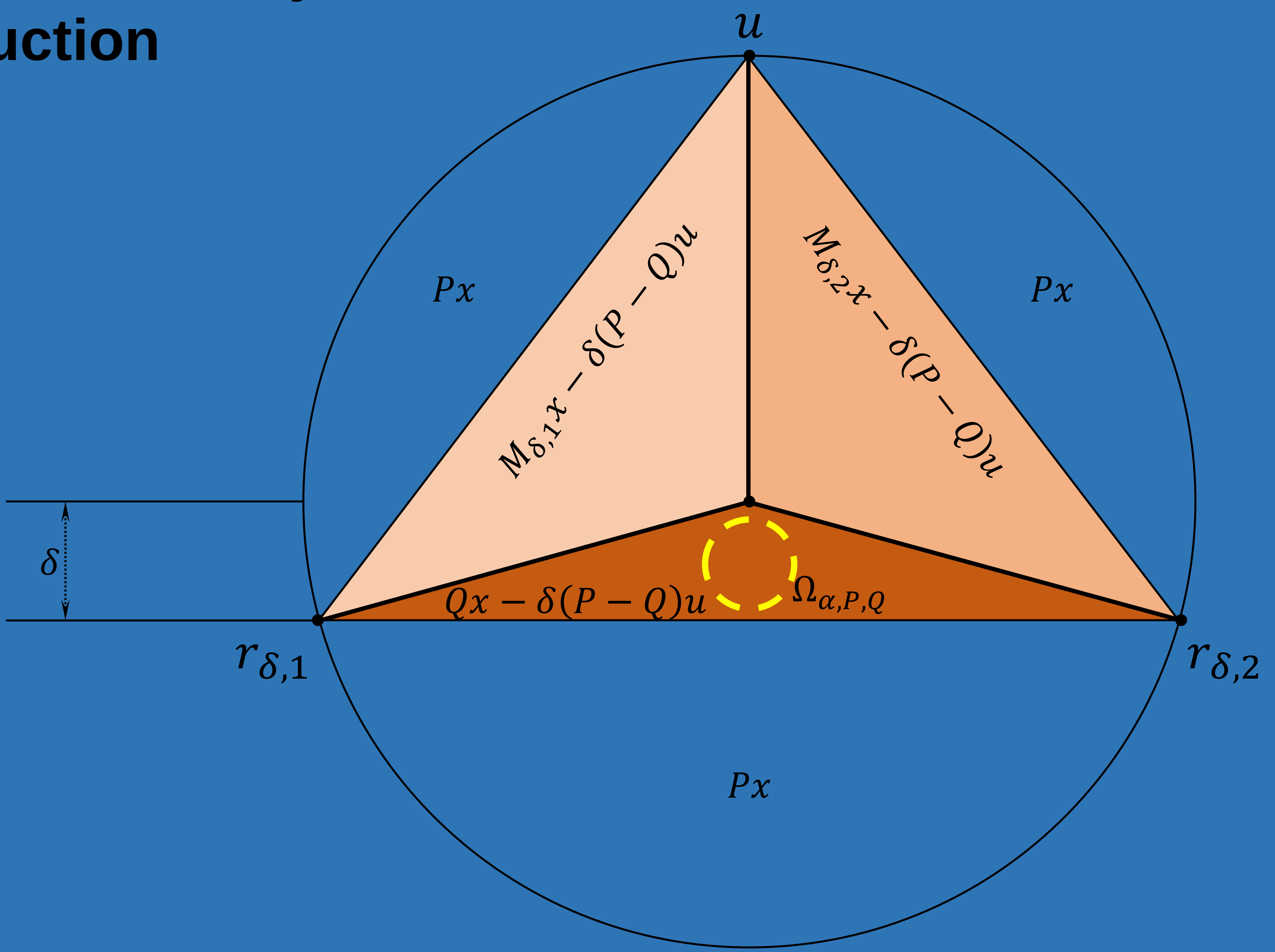
$Q \coloneqq Q_1$



The Recursive Ray-Fish Construction

$P := Q_1$

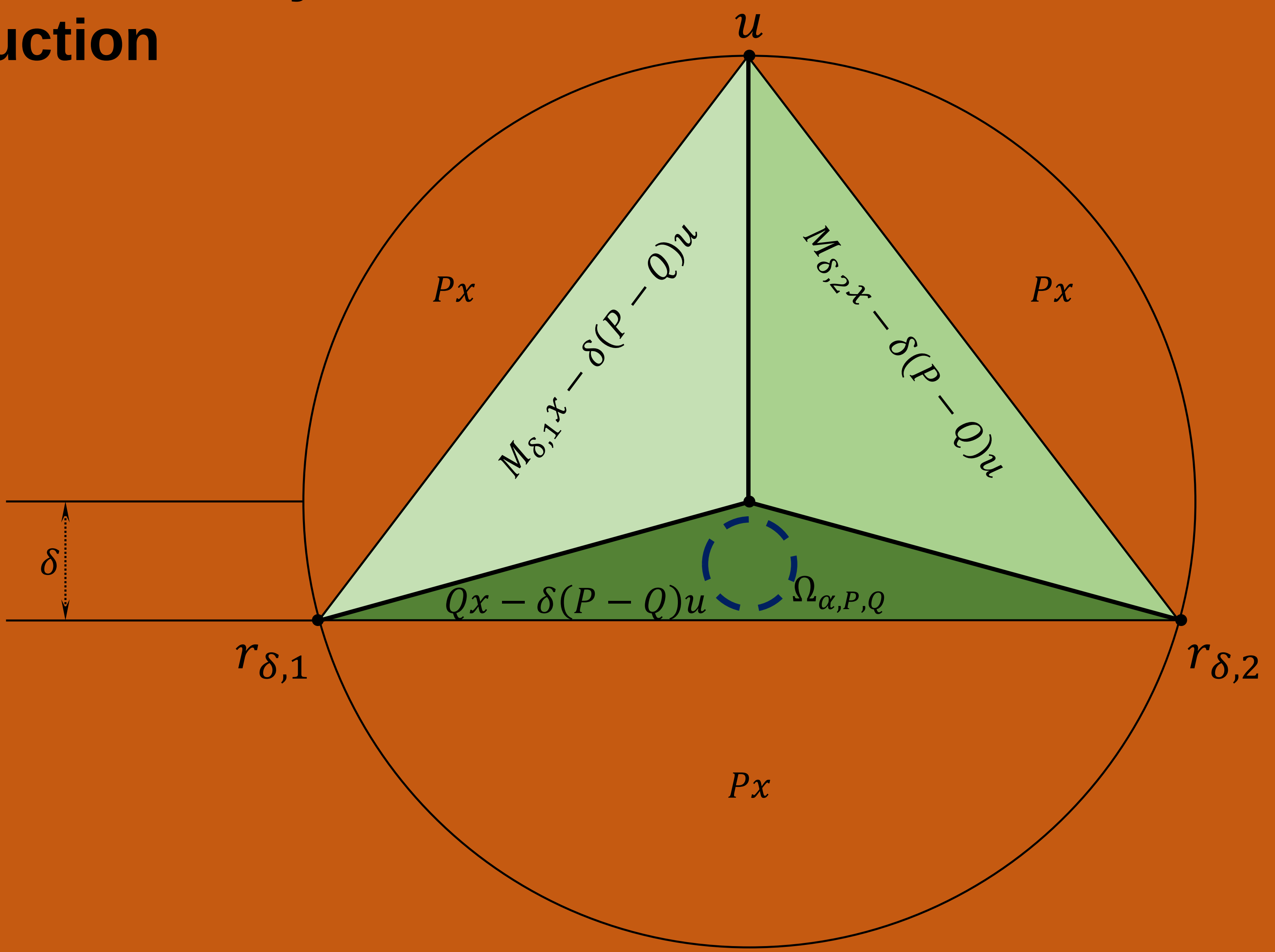
$Q := Q_2$



The Recursive Ray-Fish Construction

$P \coloneqq Q_2$

$Q \coloneqq Q_3$



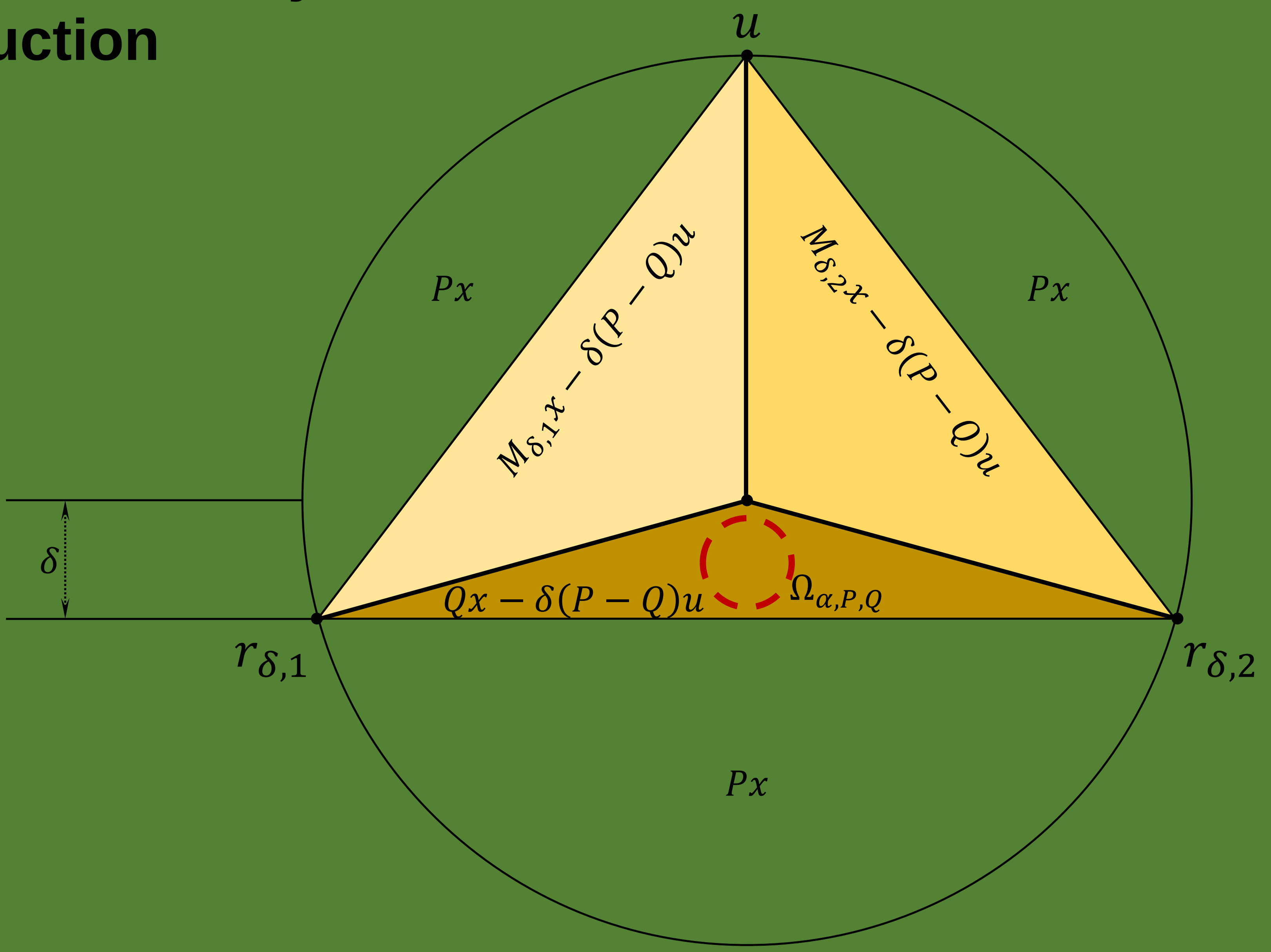
The Recursive Ray-Fish Construction

… and so on …

The Recursive Ray-Fish Construction

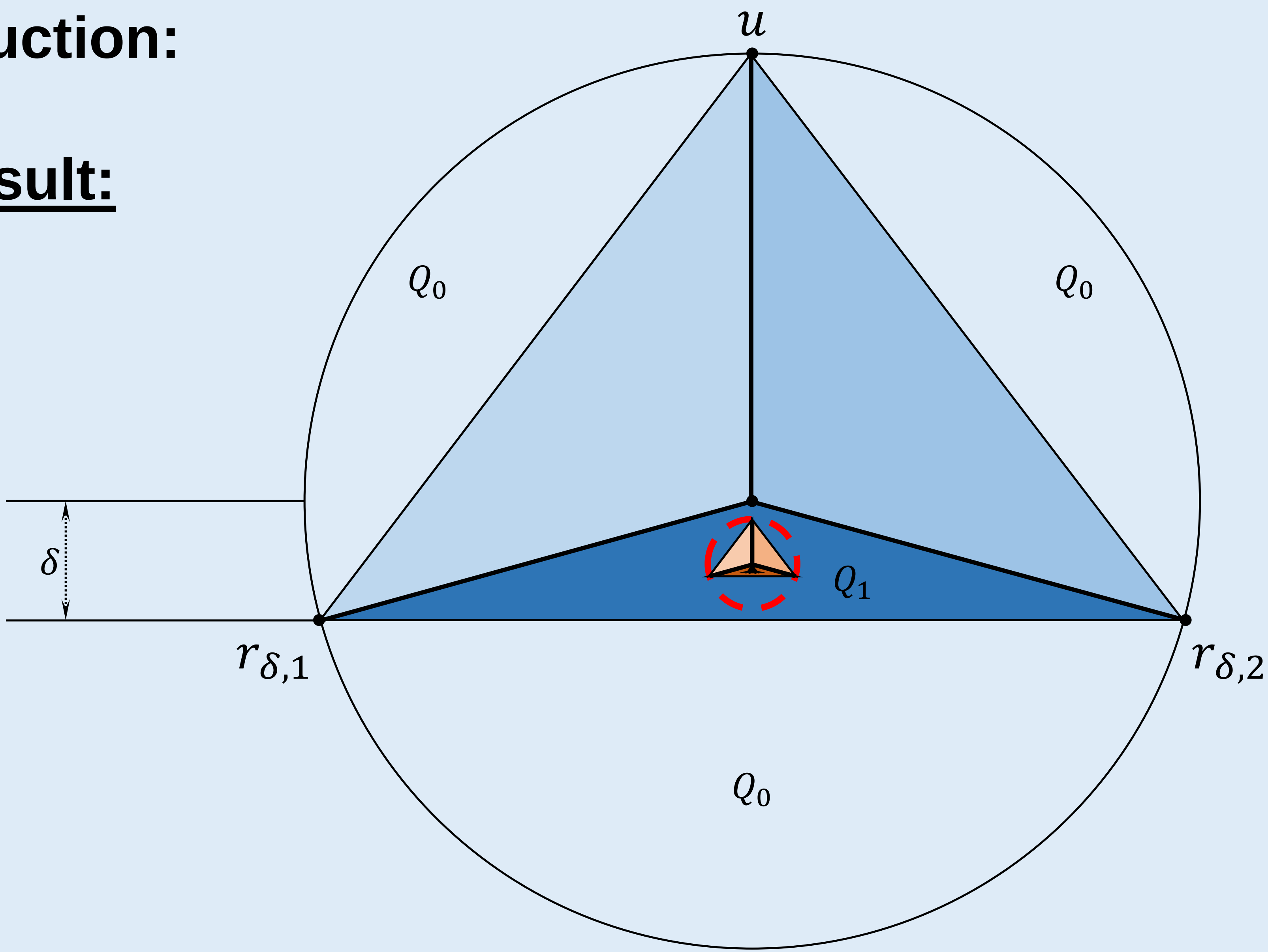
$P \coloneqq Q_{k-1}$

$Q \coloneqq Q_k$



The Recursive Ray-Fish Construction:

The Result:

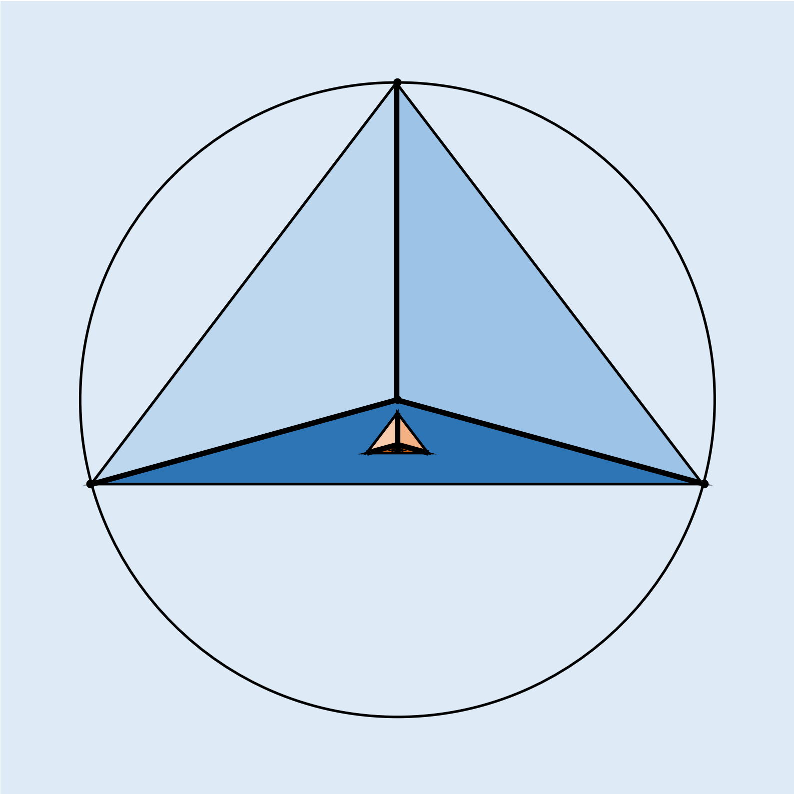


The Recursive Ray-Fish Construction: Ray-Fish Colony Lemma 2

Let $\beta > 0$ and let $Q_0, \dots, Q_k \in \mathbb{R}^{m\times n}$ be with $\mathrm{rank}\big(Q_j - Q_{j+1}\big) \leq 1$ for $j = 0, \dots, k-1$.

Recall that, as we immerse into the (shifted and scaled) open sets $\Omega_{\alpha,Q_0Q_1}, \Omega_{\alpha,Q_1Q_2}, \dots, \Omega_{\alpha,Q_{k-1}Q_k}$, we encounter all the derivatives $Q_0, Q_1, \dots, Q_k$.

Recall also that the matrices "$M_{\delta,\dots}$" tend to the matrices $Q_0, Q_1, \dots, Q_k$ as $\delta \downarrow 0$.



The Recursive Ray-Fish Construction: Ray-Fish Colony Lemma 2

Let $\beta > 0$ and let $Q_0, \ldots, Q_k \in \mathbb{R}^{m \times n}$ be with $\operatorname{rank}(Q_j - Q_{j+1}) \le 1$ for $j = 0, \ldots, k-1$.

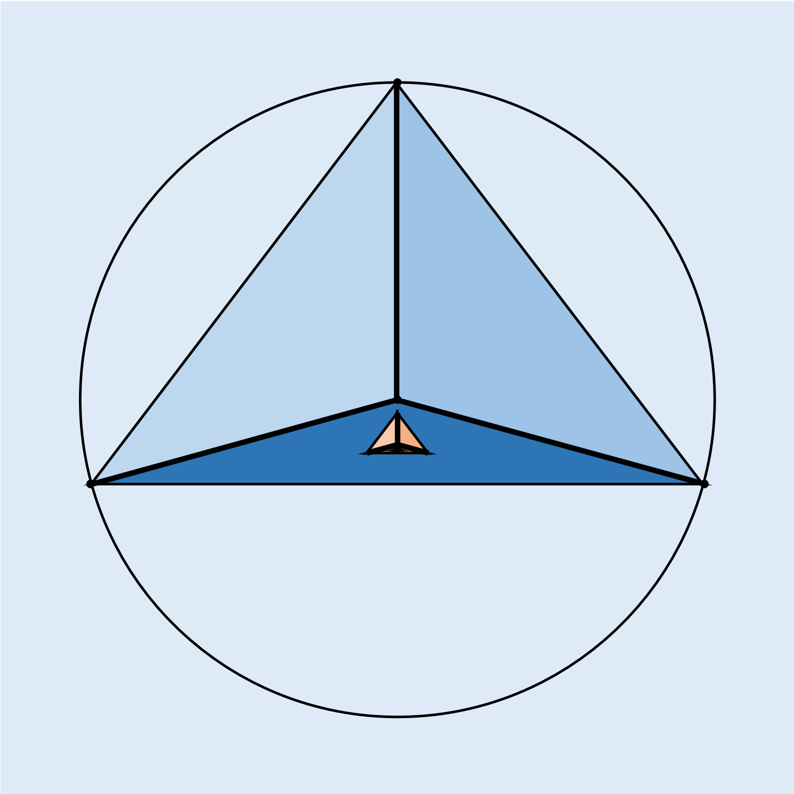
Since the matrices "$M_{\delta,\ldots}$" tend

to the matrices $Q_0, Q_1, \ldots, Q_k$ as $\delta \downarrow 0$,

it follows that

$\operatorname{dist}\left(g'_{\beta,Q_0 \ldots Q_k}(x), \{Q_0, \ldots, Q_k\}\right) < \beta$

whenever $g_{\beta,Q_0 \ldots Q_k}$ is differentiable at $x \in \mathbb{R}^n$.



Notice that here we have

$\Omega_{\beta,Q_0 \ldots Q_k} :=$ the (very small, shifted and scaled) $\Omega_{\alpha,Q_{k-1}Q_k}$

The Ray-Fish Colony for a Line Segment

Until now, we have considered matrices $P, Q \in \mathbb{R}^{m\times n}$ or $Q_0, Q_1, \dots, Q_k \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(P-Q) \leq 1$ or $\operatorname{rank}\left(Q_j - Q_{j+1}\right) \leq 1$ for $j = 0, 1, \dots, k-1$.

It is now our purpose to construct the "ray-fish colony" mapping of analogous properties for general matrices $A, B \in \mathbb{R}^{m\times n}$.

The Ray-Fish Colony for a Line Segment

Consider two matrices $U, V \in \mathbb{R}^{m \times n}$ consisting of rows $u_1, \dots, u_m \in \mathbb{R}^{1 \times n}$ and $v_1, \dots, v_m \in \mathbb{R}^{1 \times n}$, respectively.

Let T_{UV}^{i} denote the $m \times n$ matrix consisting of the first $(m - i)$ rows $u_1, \dots, u_{m-i}$ of matrix U and of the last i rows $v_{m-1+1}, \dots, v_m$ of matrix V for $i = 0, 1, \dots, m$:

$$T_{UV}^{i} = \begin{pmatrix} \cdots & u_1 & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & u_{m-i} & \cdots \\ \cdots & v_{m-i+1} & \cdots \\ \cdots & v_{m-i+2} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & v_m & \cdots \end{pmatrix} \qquad T_{UV}^{i+1} = \begin{pmatrix} \cdots & u_1 & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & u_{m-i} & \cdots \\ \cdots & u_{m-i+1} & \cdots \\ \cdots & v_{m-i+2} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & v_m & \cdots \end{pmatrix}$$

Observe that

$$\operatorname{rank}\left(T_{UV}^{i} - T_{UV}^{i+1}\right) \leq 1 \qquad \text{for} \quad i = 0, \dots, m-1$$

The Ray-Fish Colony for a Line Segment

Consider two matrices $U, V \in \mathbb{R}^{m\times n}$:

$$U = \begin{pmatrix} \cdots & u_1 & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & u_{m-i} & \cdots \\ \cdots & u_{m-i+1} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & u_m & \cdots \end{pmatrix} \qquad T_{UV}^{i} = \begin{pmatrix} \cdots & u_1 & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & u_{m-i} & \cdots \\ \cdots & v_{m-i+1} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & v_m & \cdots \end{pmatrix} \qquad V = \begin{pmatrix} \cdots & v_1 & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & v_{m-i} & \cdots \\ \cdots & v_{m-i+1} & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & v_m & \cdots \end{pmatrix}$$

Observe also that

$$T_{UV}^{0} = U \qquad \text{and} \qquad T_{UV}^{m} = V$$

and

$$\max\{\|T_{UV}^{i} - U\|, \|T_{UV}^{i} - V\|\} \leq \|U - V\| \qquad \text{for} \quad i = 0, 1, \ldots, m$$

The Ray-Fish Colony for a Line Segment

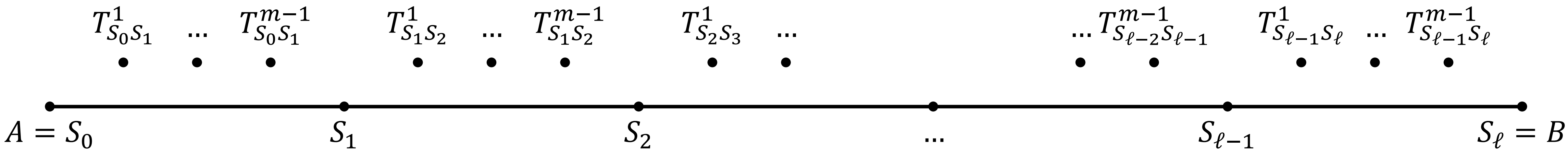
Consider any matrices $A, B \in \mathbb{R}^{m\times n}$.

Recall that the line segment between the matrices A and B is the convex hull

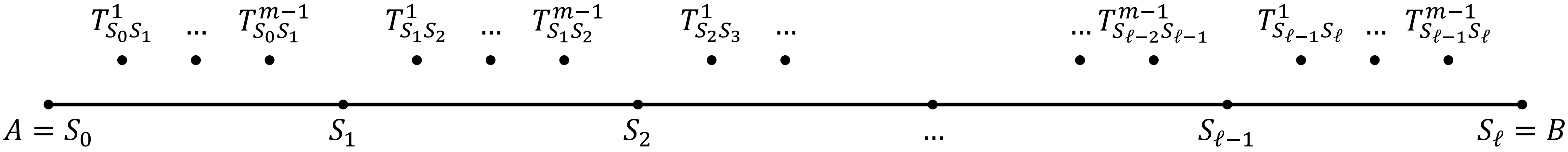
$$[A, B] \coloneqq \mathrm{co}\{A, B\}$$

Considering a positive natural number ℓ,

divide the line segment by $(\ell - 1)$ points $S_1, \dots, S_{\ell-1}$ equidistantly:



The Ray-Fish Colony for a Line Segment



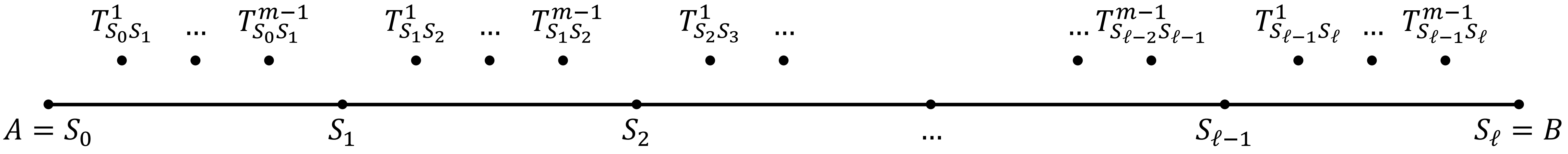
Notice that

$$\operatorname{dist}\left(T^i_{S_jS_{j+1}},\ [A,B]\right) \le \max\left\{\left\|T^i_{S_jS_{j+1}} - S_j\right\|,\ \left\|T^i_{S_jS_{j+1}} - S_{j+1}\right\|\right\} \le$$

$$\le \left\|S_j - S_{j+1}\right\| = \frac{1}{\ell}\|A - B\| \downarrow 0 \qquad \text{as} \quad \ell \to \infty$$

The Ray-Fish Colony for a Line Segment

Rename this cortege of matrices



to

Q_0 Q_1 … Q_{m-1} Q_m … Q_{2m} … … … $Q_{(\ell-1)m}$ … $Q_{\ell m}$

Recall that

$$\operatorname{dist}\big(Q_j,\ [A,B]\big) \le \frac{1}{\ell}\|A-B\| \ \downarrow\ 0 \qquad \text{as} \quad \ell \to \infty$$

and notice that

$$\operatorname{rank}\big(Q_j - Q_{j+1}\big) \le 1 \qquad \text{for} \quad j = 0,\ \dots,\ \ell m - 1$$

By applying the Ray-Fish Colony Construction, we obtain:

Ray-Fish Colony for a Line Segment: Lemma 3

Let $\gamma > 0$ and let $A, B \in \mathbb{R}^{m\times n}$ be any matrices.

Then there exists a finitely piecewise affine Lipschitzian mapping $h_{\gamma,[A,B]}: \mathbb{R}^n \to \mathbb{R}^m$

such that

- $h_{\gamma,[A,B]}(x) = Ax$ for all $x \in \mathbb{R}^n \setminus \mathbb{B}_n$,
- $h'_{\gamma,[A,B]}(x) = B$ for all x from a non-empty open set $\Omega_{\gamma,[A,B]} \subset \mathbb{B}_n$,
- and

$$\operatorname{dist}\left(h'_{\gamma,[A,B]}(x),\ [A,B]\right) < \gamma$$

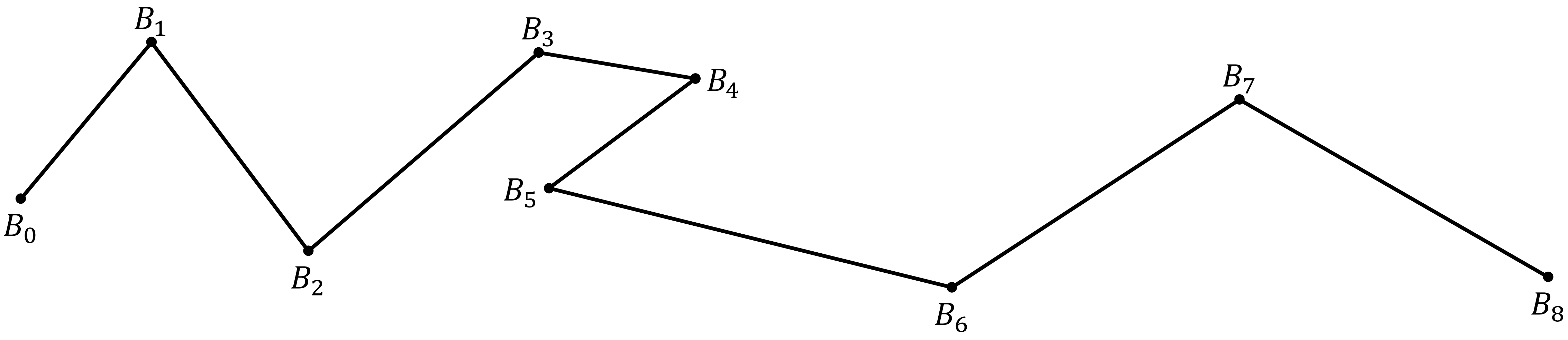
whenever $h_{\gamma,[A,B]}$ is differentiable at $x \in \mathbb{R}^n$.

The Ray-Fish Colony for a Polygonal Chain

Consider any matrices $B_0, B_1, \dots, B_N \in \mathbb{R}^{m\times n}$.

Recall that the polygonal chain $[B_0, B_1, \dots, B_N]$ is a curve which consists of the line segments connecting the consecutive vertices, that is the union of the convex hulls

$$[B_0, B_1, \dots, B_N] := \mathrm{co}\{B_0, B_1\} \cup \mathrm{co}\{B_1, B_2\} \cup \cdots \cup \mathrm{co}\{B_{N-1}, B_N\}$$



The Ray-Fish Colony for a Polygonal Chain

Given the matrices $B_0, B_1, \dots, B_N \in \mathbb{R}^{m\times n}$, consider a positive natural ℓ,

- divide each of the line segments $[B_0, B_1]$, $[B_1, B_2]$, …, $[B_{N-1}, B_N]$

  by $(\ell - 1)$ points $S_1^1, \dots, S_{\ell-1}^1$, $S_1^2, \dots, S_{\ell-1}^2$, …, $S_1^N, \dots, S_{\ell-1}^N$, equidistantly,

- consider also the intervening transitional matrices:

$$B_0 = S_0^1 = T^0_{S_0^1 S_1^1}, \dots, T^m_{S_0^1 S_1^1} = S_1^1 = T^0_{S_1^1 S_2^1}, \dots, T^m_{S_1^1 S_2^1} = S_2^1 = T^0_{S_2^1 S_3^1}, \quad \dots, \quad T^m_{S_{\ell-1}^1 S_\ell^1} = S_\ell^1 = B_1$$

$$B_1 = S_0^2 = T^0_{S_0^2 S_1^2}, \dots, T^m_{S_0^2 S_1^2} = S_1^2 = T^0_{S_1^2 S_2^2}, \dots, T^m_{S_1^2 S_2^2} = S_2^2 = T^0_{S_2^2 S_3^2}, \quad \dots, \quad T^m_{S_{\ell-1}^2 S_\ell^2} = S_\ell^2 = B_2$$

...

$$B_{N-1} = S_0^N = T^0_{S_0^N S_1^N}, \dots, T^m_{S_0^N S_1^N} = S_1^N = T^0_{S_1^N S_2^N}, \dots, T^m_{S_1^N S_2^N} = S_2^N = T^0_{S_2^N S_3^N}, \quad \dots, \quad T^m_{S_{\ell-1}^N S_\ell^N} = S_\ell^N = B_N$$

- and rename this long cortege of matrices to $Q_0, \dots, Q_{\ell m N}$.

The Ray-Fish Colony for a Polygonal Chain

We then have:

$$\mathrm{dist}\big(Q_j,\ [B_0, B_1, \ldots, B_N]\big) \le \frac{1}{\ell}\max\{\|B_0 - B_1\|,\ \ldots,\ \|B_{N-1} - B_N\|\}\ \downarrow\ 0 \qquad \text{as} \quad \ell \to \infty$$

Notice also that

$$\mathrm{rank}\big(Q_j - Q_{j+1}\big) \le 1 \qquad \text{for} \quad j = 0,\ \ldots,\ \ell m N - 1$$

By applying the Ray-Fish Colony Construction, we obtain:

The Ray-Fish Colony for a Polygonal Chain: Lemma 4

Let $\gamma > 0$ and let $B_0, B_1, \ldots, B_N \in \mathbb{R}^{m \times n}$ be any matrices.

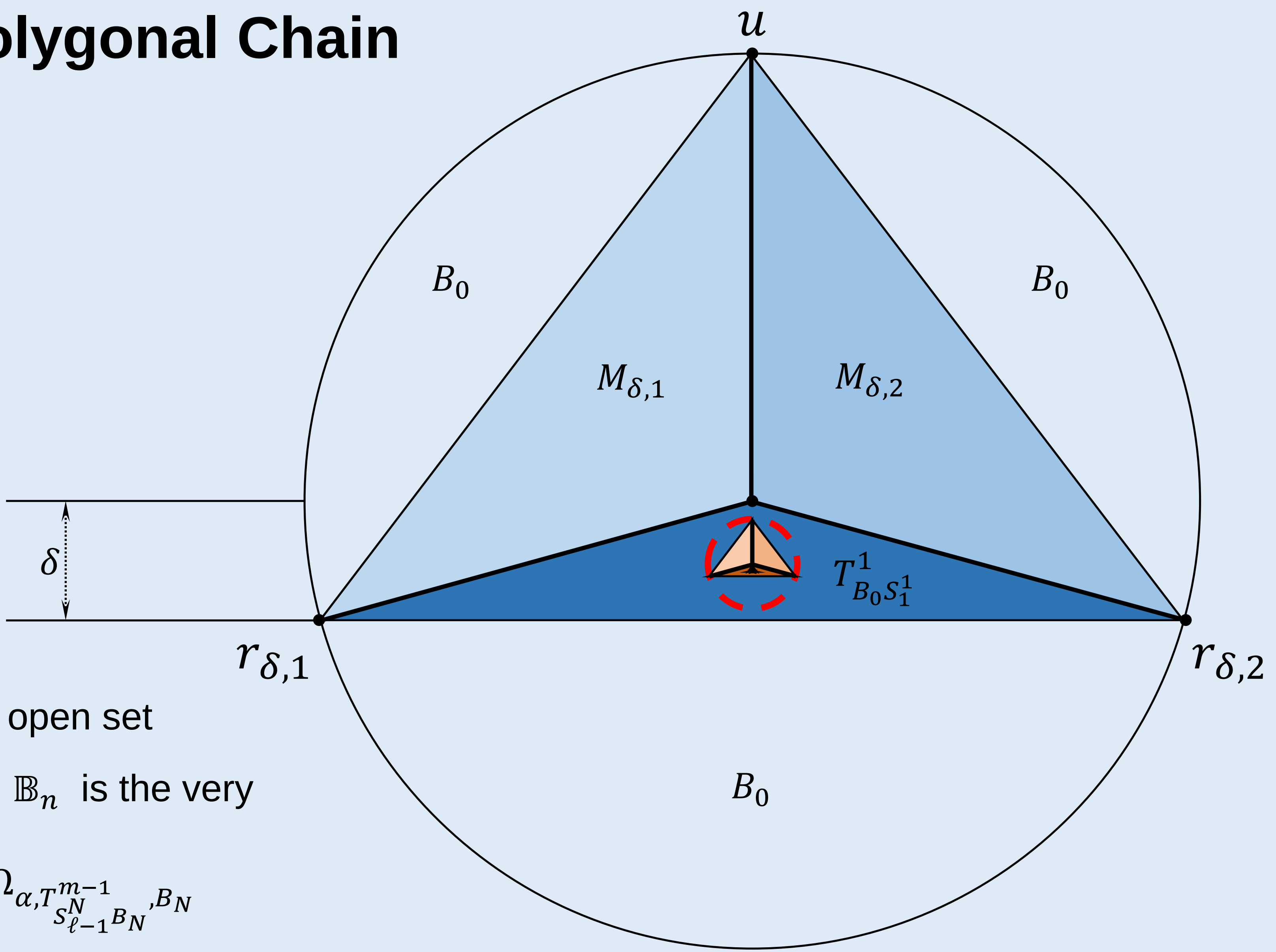
Then there exists a Lipschitzian mapping $h_{\gamma,[B_0,B_1,\ldots,B_N]}: \mathbb{R}^n \to \mathbb{R}^m$ such that it is finitely piecewise affine and

- $h_{\gamma,[B_0,B_1,\ldots,B_N]}(x) = B_0 x$ for all $x \in \mathbb{R}^n \setminus \mathbb{B}_n$,
- $h'_{\gamma,[B_0,B_1,\ldots,B_N]}(x) = B_N$ for all x from a non-empty open set $\Omega_{\gamma,[B_0,B_1,\ldots,B_N]} \subset \mathbb{B}_n$,
- and

$$\operatorname{dist}\left(h'_{\gamma,[B_0,B_1,\ldots,B_N]}(x),\ [B_0, B_1, \ldots, B_N]\right) < \gamma$$

whenever $h_{\gamma,[B_0,B_1,\ldots,B_N]}$ is differentiable at $x \in \mathbb{R}^n$.

Ray-Fish Colony Construction for a Polygonal Chain



Notice that the open set

$\Omega_{\gamma,[B_0,B_1,\ldots,B_N]} \subset \mathbb{B}_n$ is the very

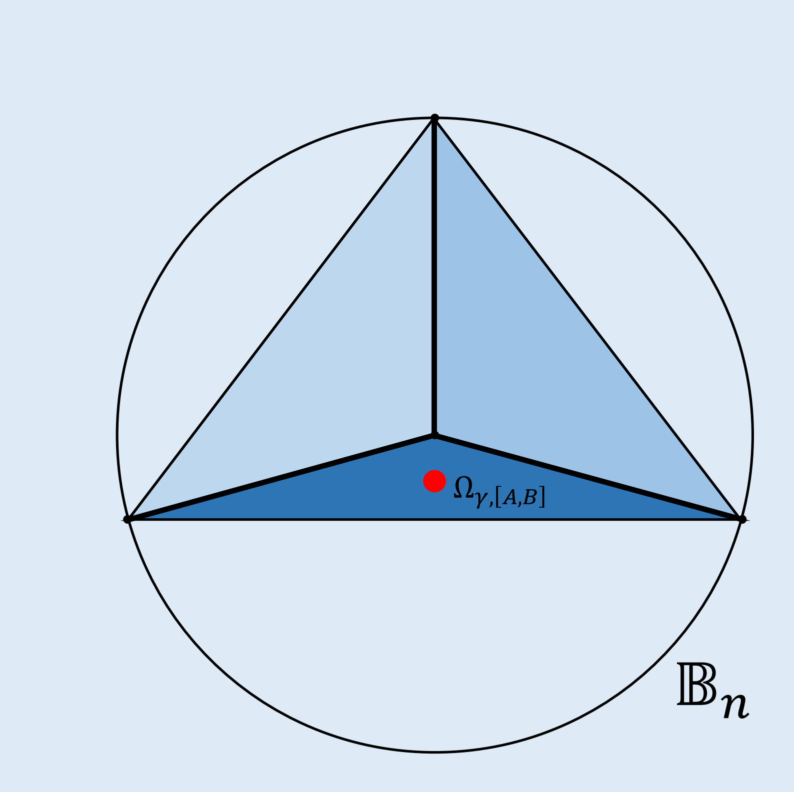
last open set $\Omega_{\alpha,T^{m-1}_{S^N_{\ell-1}B_N},B_N}$

inside the very last ray-fish inside.

The Corona Construction

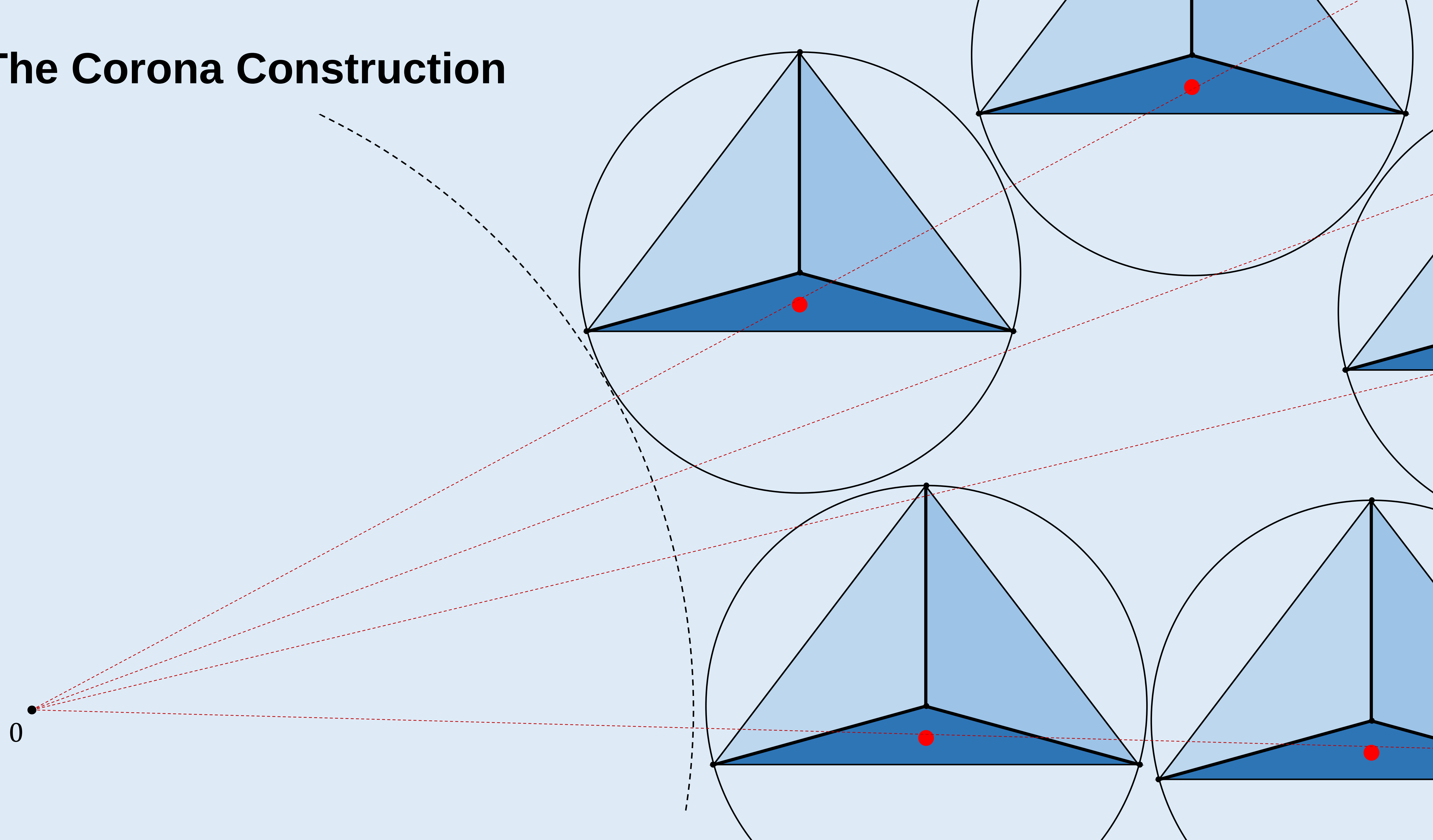
For a $\gamma > 0$ and for $B_0, B_1, \dots, B_N \in \mathbb{R}^{m\times n}$, we have constructed (recursively) the Ray-Fish Colony $h_{\gamma,[B_0,B_1,\dots,B_N]}\colon \mathbb{R}^n \to \mathbb{R}^m$ for the polygonal chain $[B_0, B_1, \dots, B_N]$.

The non-empty open set $\Omega_{\gamma,[B_0,B_1,\dots,B_N]} \subset \mathbb{B}_n$ is very small:



Now, make many copies of this ray-fish colony $h_{\gamma,[B_0,B_1,\dots,B_N]}$, shift them and place them beyond the sphere of radius 3, say, in such a way that each ray emanating from the origin passes through at least one of the (shifted) open sets $\Omega_{\gamma,[B_0,B_1,\dots,B_N]}$, the colonies being pairwise disjoint.

The Corona Construction



The Corona Construction

Plenty (finitely many) of copies of the

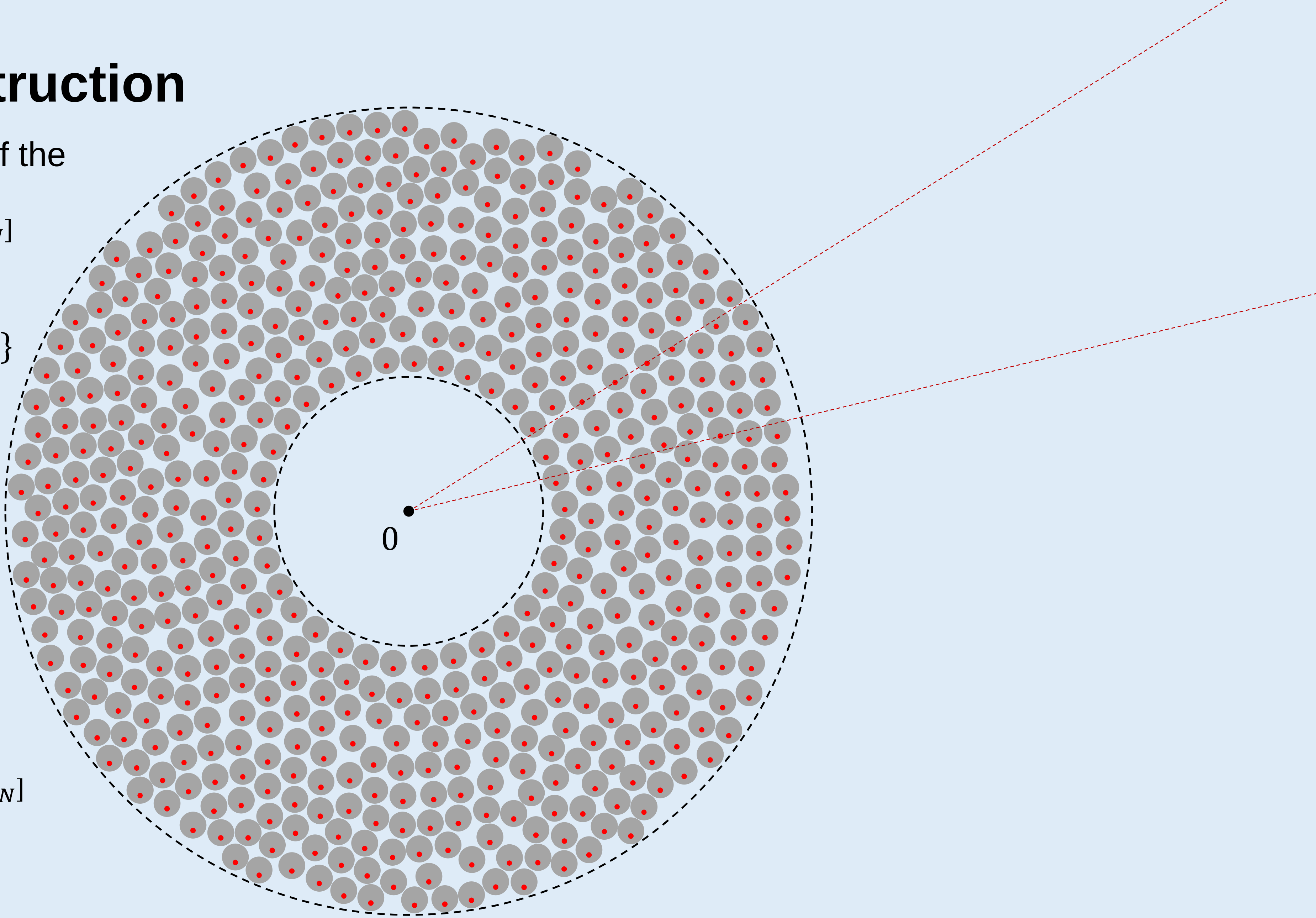
ray-fish colony "●" = $h_{\gamma,[B_0,B_1,\ldots,B_N]}$

are placed into a spherical shell

$T(\rho, P) = \{ x \in \mathbb{R}^n : \rho \leq \|x\| \leq P \}$

in such a way that

- the colonies are pairwise disjoint, and
- each ray emanating from the origin passes through at least one of the (shifted) tiny open set "•" = $\Omega_{\gamma,[B_0,B_1,\ldots,B_N]}$



for $0 < \rho < P < +\infty$.

We thus obtain:

Ray-Fish Corona Construction for a Polygonal Chain: Lemma 5

Let $\delta > 0$ and let $B_0, B_1, \dots, B_N \in \mathbb{R}^{m\times n}$ be any matrices.

Then there exist numbers $0 < \rho_{\delta,[B_0,B_1,\dots,B_N]} < P_{\delta,[B_0,B_1,\dots,B_N]} < +\infty$ and

a finitely piecewise affine Lipschitzian mapping $\varphi_{\delta,[B_0,B_1,\dots,B_N]}\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

- $\varphi_{\delta,[B_0,B_1,\dots,B_N]}(x) = B_0 x$ for all $x \in \mathbb{R}^n \setminus T\big(\rho_{\delta,[B_0,B_1,\dots,B_N]}, P_{\delta,[B_0,B_1,\dots,B_N]}\big)$,
- for every subspace $\{0\} \not\subseteq W \subset \mathbb{R}^n$, there are a $w \in W$ and a $\lambda > 0$ such that

  the ball $B(w,\lambda) \subset T\big(\rho_{\delta,[B_0,B_1,\dots,B_N]}, P_{\delta,[B_0,B_1,\dots,B_N]}\big)$ and

  $\varphi'_{\delta,[B_0,B_1,\dots,B_N]}(x) = B_N$ for all $x \in W \cap B(w,\lambda)$,
- and

$$\mathrm{dist}\big(\varphi'_{\delta,[B_0,B_1,\dots,B_N]}(x), [B_0, B_1, \dots, B_N]\big) < \delta$$

  whenever $\varphi_{\delta,[B_0,B_1,\dots,B_N]}$ is differentiable at $x \in \mathbb{R}^n$.

Finale

Finale

Given a non-empty compact convex set $\mathcal{P} \subset \mathbb{R}^{m\times n}$ of matrices,

it is our purpose to construct a Lipschitzian mapping $g\colon \mathbb{R}^n \to \mathbb{R}^m$

such that

$$\partial g(0) = \mathcal{P}$$

actually

$$\partial g_{|W}(0) = \mathcal{P}_{|W} \qquad \text{for every linear subspace} \quad \{0\} \nsubseteq W \subset \mathbb{R}^n$$

Finale

The non-empty compact convex set $\mathcal{P} \subset \mathbb{R}^{m \times n}$ of matrices is separable.

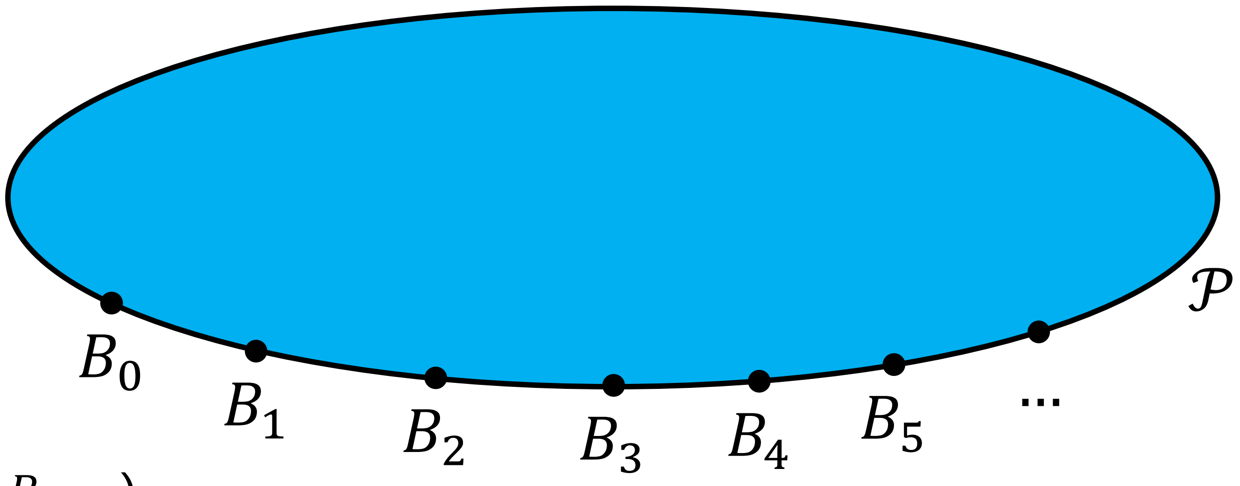
Therefore, there exists a countable sequence

$$B_0, B_1, B_2, B_3, B_4, B_5, \ldots \in \mathcal{P}$$

such that the convex hull

$$\mathrm{co}\{B_0, B_1, B_2, B_3, B_4, B_5, \ldots\}$$

of the set is dense in $\mathcal{P}$.



(Remark: If already $\mathrm{co}\{B_0, B_1, \ldots, B_N\}$ is dense in $\mathcal{P}$, then consider $B_0, B_1, \ldots, B_N, B_0, B_1, \ldots, B_N, B_0, B_1, \ldots, B_N, \ldots$)

Finale

Given the non-empty compact convex set $\mathcal{P} \subset \mathbb{R}^{m \times n}$ of matrices and having the countably infinite sequence $B_0, B_1, B_2, B_3, B_4, B_5, \ldots \in \mathcal{P}$, consider the longer and longer polygonal chains

$$[B_0, B_1]$$

$$[B_0, B_1, B_2]$$

$$[B_0, B_1, B_2, B_3]$$

$$[B_0, B_1, B_2, B_3, B_4]$$

$$[B_0, B_1, B_2, B_3, B_4, B_5]$$

and so on

Finale

Given the non-empty compact convex set $\mathcal{P} \subset \mathbb{R}^{m \times n}$ of matrices and having polygonal chains $[B_0, B_1, \ldots, B_N]$ for each $N \in \mathbb{N}$, consider also a decreasing sequence

$$\delta_1 > \delta_2 > \delta_3 > \delta_4 > \delta_5 > \cdots > 0$$

such that

$$\delta_N \downarrow 0 \qquad \text{as} \quad N \to \infty$$

Finale

Given the non-empty compact convex set $\mathcal{P} \subset \mathbb{R}^{m\times n}$ of matrices,

having polygonal chains $[B_0, B_1, \dots, B_N]$ for each $N \in \mathbb{N}$,

and having also the decreasing sequence $\delta_N \downarrow 0$, with $N \in \mathbb{N}$,

construct the Coronas for these polygonal chains with these “delta’s”:

$$\varphi_{\delta_1,[B_0,B_1]}$$

$$\varphi_{\delta_2,[B_0,B_1,B_2]}$$

$$\varphi_{\delta_3,[B_0,B_1,B_2,B_3]}$$

$$\varphi_{\delta_4,[B_0,B_1,B_2,B_3,B_4]}$$

$$\varphi_{\delta_5,[B_0,B_1,B_2,B_3,B_4,B_5]}$$

and so on

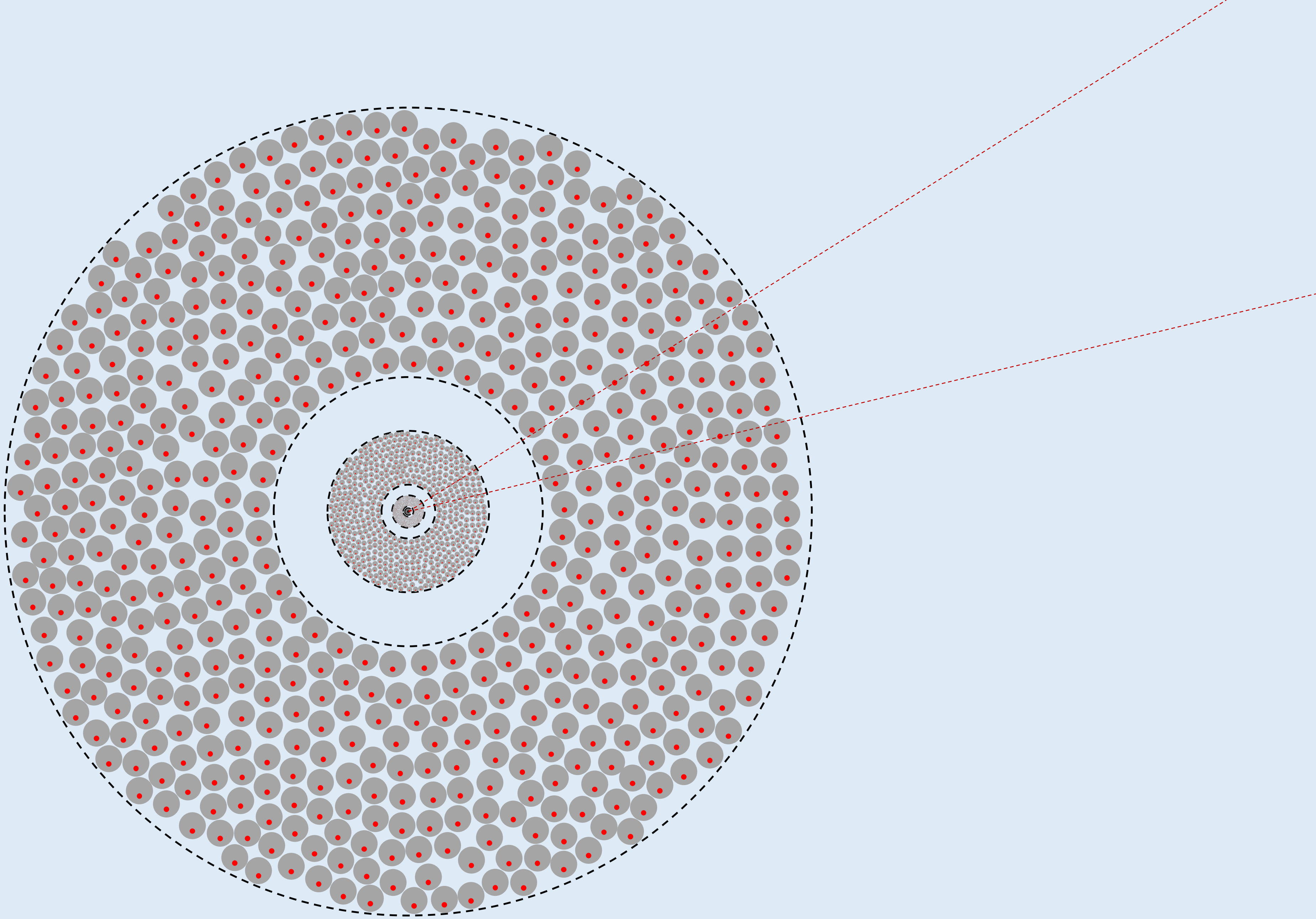
Finale

Given the non-empty compact convex set $\mathcal{P} \subset \mathbb{R}^{m \times n}$ of matrices,

- take the first corona $\varphi_{\delta_1,[B_0,B_1]}$,
- take the second corona $\varphi_{\delta_2,[B_0,B_1,B_2]}$ and shrink it to be inside the first one
- take the third corona $\varphi_{\delta_3,[B_0,B_1,B_2,B_3]}$ and shrink it to be inside the second one
- take the fourth corona $\varphi_{\delta_4,[B_0,B_1,B_2,B_3,B_4]}$ and shrink it to be inside the third one
- and so on

That is, the coronas tend to the origin.

Finale



We have thus obtained:

Finale: The Main Result: Theorem

Let $m, n \in \mathbb{N}$ and let $\mathcal{P} \subset \mathbb{R}^{m \times n}$ be any non-empty compact convex set of matrices.

Then there exists a countably piecewise affine Lipschitzian mapping $g: \mathbb{R}^n \longrightarrow \mathbb{R}^m$,

with $g(0) = 0$, such that, for every linear subspace $\{0\} \subsetneq W \subset \mathbb{R}^n$,

the Clarke Jacobian

$$\partial g_{|W}(0) = \mathcal{P}_{|W}$$

References

- D. Bartl, M. Fabian, *Every compact convex subset of matrices is the Clarke Jacobian of some Lipschitzian mapping*, Proceedings of the American Mathematical Society, in press.

- D. Bartl, M. Fabian, *Can Pourciau's open mapping theorem be derived from Clarke's inverse mapping theorem easily?*, Journal of Mathematical Analysis and Applications 497(2) (2021) 124858.

Thank You
for your attention