Variational Analysis and Optimisation Webinars - March 31, 2021

Characterizing quasiconvexity of the pointwise infimum of a family of arbitrary translations of quasiconvex functions *

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Introduction

The Framework:

X : real Banach space, X^* : continuous dual, $\langle \cdot, \cdot \rangle$ the pairing between X and X^*. For $f : X \to \mathbb{R} \cup \{+\infty\}$,

- $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ (domain).
- $\operatorname{epi}(f) = \{(x, \alpha) \in X \times \mathbb{R} : f(x) \leq \alpha\}$ (epigraph).
- For any $\lambda \in \mathbb{R}$

  $[f \leq \lambda] \doteq \{x \in X : f(x) \leq \lambda\}$ (*sublevel set at height* λ)
  $[f < \lambda] \doteq \{x \in X : f(x) < \lambda\}$ (*strict sublevel set at height* λ)

Motivation: convex functions vs quasiconvex functions

f convex $\Leftrightarrow$ Epi(f) is convex.

f quasiconvex $\Leftrightarrow S_\lambda$ is convex, $\forall\, \lambda \in \mathbb{R}$.

- If f, g convex $\Rightarrow f + g$ convex.
- If f, g quasiconvex $\Rightarrow f + g$ is not in general quasiconvex.

Example

$f(x) = x^2$, $g(x) = -x^3 \Rightarrow (f + g)(x) = x^2 - x^3$.

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Characterization subdifferential

- f convex: (classical) Subdifferential

  f is a lcs convex $\Leftrightarrow \partial f$ maximal monotone.

- f lsc: Clarke-Rockafellar subdifferential:

$$\partial f(x) = \{x^* \in X : f^{\uparrow}(x,u) \geq \langle x^*, u\rangle, \forall\, u \in X\Big\},\ x \in \mathrm{dom} f.[6]$$

f quasiconvex lcs $\Leftrightarrow \partial f$ quasimonotone. [3]

- $T : X \rightrightarrows X^*$ monotone, $x^* \in X^* \Rightarrow T + x*$ monotone.
- [3] $T : X \rightrightarrows X^*$ quasimonotone $\Rightarrow$

  $[\forall\, x^* \in X^*, T + x^*$ quasimonotone $\Leftrightarrow T$ monotone.]

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- [3] $T : X \rightrightarrows X^*$ quasimonotone $\Rightarrow$

$$[\forall\, x^* \in X^*, T + x^* \text{ quasimonotone } \Leftrightarrow T \text{ monotone.}]$$

How do we preserve quasi-convexity under summation?

A natural case: $f : X \to \mathbb{R} \cup \{+\infty\}$ quasiconvex and $g : \mathbb{R} \to \mathbb{R}$ nondecreasing.

$$[g \circ f \leq \lambda], \text{ is convex } \forall \lambda \in \mathbb{R}$$

so, $g \circ f$ quasiconvex and it is easy to show that $f + g \circ f$ is quasiconvex.

$$g \circ f + f \text{ is quasiconvex.}$$

in other cases?

Quasiconvex and quasimonotone families

Definition

A family $\mathcal{A}$ of operators $T_i : X \rightrightarrows X^*$, $i \in I$, will be called a quasimonotone family, if the operator T with graph $\mathrm{Gr}\, T = \bigcup_{i\in I} \mathrm{Gr}\, T_i$ is quasimonotone.

Two operators T_1, T_2 will be called a quasimonotone pair if $\{T_1, T_2\}$ is a quasimonotone family.

Definition

A family of functions $f_i : X \to \mathbb{R} \cup \{+\infty\}$, $i \in I$, is called a *quasiconvex family* if for every $i, j \in I$ and every $x, y \in X$, $z \in]x, y[$, the following implication holds:

$$f_i(x) < f_i(z) \Rightarrow f_j(z) \leq f_j(y).$$

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First result

Our first main result relates quasiconvexity of a pair of functions to quasi- monotonicity of the pair of subdifferentials.

Theorem

Let $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$ *be lsc functions. Then* $\{f_1, f_2\}$ *is a quasiconvex pair if and only if* $\{\partial f_1, \partial f_2\}$ *is a quasimonotone pair.*

Tools:

[[2, Corollary 4.3]]

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc function, and $a, b \in X$ with $f(a) < f(b)$. Then there exist $c \in [a, b[$ and sequences $x_n \to c$ and $x_n^* \in \partial f(x_n)$, such that $f(x_n) \to f(c)$ and $\langle x_n^*, x - x_n \rangle > 0$, for every $x = c + t(b - a)$ with $t > 0$.

[[1, Theorem 2.1]]

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lsc function. The following are equivalent:

(i) f is quasiconvex;

(ii) $\exists\, x^* \in \partial f(x) : \langle x^*, y - x \rangle > 0 \Rightarrow f(z) \leq f(y), \forall z \in [x, y]$.

How big is the class of quasiconvex pairs of lsc functions

Candidates:

Type 1: f_1 and f_2 are quasiconvex, and there is a proportionality between the subdifferentials: $\partial f_i(x) \subseteq \mathbb{R}_+ \partial f_j(x)$,$\forall\, x \in X$, $i \neq j$,$i = 1$ or $i = 2$.

Type 2: f_1 and f_2 are nondecreasing transformations of a same quasiconvex function; that is, there exists a quasiconvex function g and nondecreasing functions h_1, h_2 such that $f_1 = h_1 \circ g$ and $f_2 = h_2 \circ g$.

Type 3: f_1,f_2 quasiconvex with $\operatorname{argmin}\, f_1 \cap \operatorname{argmin}\, f_2 \neq \emptyset$.

Examples:

Type 3

- $X = \mathbb{R}^2$;

- $f_1(x_1, x_2) = \min\left\{100x_1^2 + x_2^2, 1\right\}$;
- $f_2(x_1, x_2) = \min\left\{x_1^2 + 100x_2^2, 1\right\}$.
- $\operatorname{argmin} f_1 = \operatorname{argmin} f_2 = \{0\}$.
- $f = f_1 + f_2$: $f\,(0.8, 0) = f\,(0, 0.8) = 1.64$; $f\,(0.4, 0.4) = 2$

$\Rightarrow f$ is not quasiconvex.

Neither Type 1 nor Type 2

- $X = \mathbb{R}$;

$f_1(x) = \begin{cases} x^2, x < 1, \\ 1, x \geq 1 \end{cases}$; $f_2(x) = f_1\,(-x)$.

$\partial f_1 \cup \partial f_2$ is quasimonotone;

Neither $\mathbb{R}_+ \partial f_1(x) \subseteq \mathbb{R}_+ \partial f_2(x)$, $\forall\ x \in \operatorname{Dom} \partial f_2$, nor $\mathbb{R}_+ \partial f_2(x) \subseteq \mathbb{R}_+ \partial f_1(x)$, $\forall x \in \operatorname{Dom} \partial f_1$ hold.

$f_1 + f_2$ is quasiconvex.

Characterizations of quasiconvexity of the minimum of any vertical translation of two quasiconvex functions.

Theorem

Assume that the functions f_1, f_2 are quasiconvex. Then, the following assertions are equivalent:

(a) $\{f_1, f_2\}$ *is a quasiconvex pair.*

(b) *for every* $\lambda_1, \lambda_2 \in \mathbb{R}$, $[f_1 \leq \lambda_1] \cup [f_2 \leq \lambda_2]$ *is convex.*

(c) *for every* $\lambda_1, \lambda_2 \in \mathbb{R}$, $[f_1 < \lambda_1] \cup [f_2 < \lambda_2]$ *is convex.*

(d) *for every* $x \in X$, $[f_1 < f_1(x)] \cup [f_2 < f_2(x)]$ *is convex.*

(e) *for every* $\alpha \in \mathbb{R}$, *the function* h_α *defined by* $h_\alpha(x) \doteq \min\{f_1(x) + \alpha, f_2(x)\}$ *is quasiconvex.*

* *Equivalences* $(a) - (e)$ *are true in any vector space*

Example

Here, $\operatorname{argmin} f_1 \cap \operatorname{argmin} f_2 = \emptyset$. Simply consider $f_1(x) = \min\{0, x\}$ and $f_2(x) = \max\{0, x\}$. The subdifferentials are a quasimonotone pair, whereas one function has no minima, the other does have minima. It is obvious that $f_1 + f_2$ and $\min\{f_1, f_2\}$ are quasiconvex.

Example

Consider the functions f_1, f_2 defined on $\mathbb{R}^2$ by

$$f_1(x_1, x_2) = \begin{cases} \max\{\arctan x_1, 0\} & \text{if} \quad -1 \le x_2 \le +1 \\ \frac{\pi}{2} & \text{if} \quad x_2 < -1 \\ x_2 - 1 + \frac{\pi}{2} & \text{if} \quad x_2 > 1 \end{cases}$$

$$f_2(x_1, x_2) = \begin{cases} \max\{-\arctan x_1, 0\} & \text{if} \quad -1 \le x_2 \le +1 \\ -x_2 - 1 + \frac{\pi}{2} & \text{if} \quad x_2 < -1 \\ \frac{\pi}{2} & \text{if} \quad x_2 > 1 \end{cases}$$

One may check that the union of any two sublevel sets is convex, so the functions are a quasiconvex pair $\Rightarrow \min\{f_1, f_2\}$ is quasiconvex.

Some consequences for the sum of quasiconvex functions

Let $J \doteq \{1, 2, \dots, m\}$.

Theorem

Let $\{f_i : i \in J\}$ be a quasiconvex family. Then $f_1 + f_2 + \cdots + f_m$ and $\min\{f_1, f_2, \cdots, f_n\}$ are quasiconvex.

Another characterization:

Proposition

Let f_1, f_2 be functions on X. Then $\{f_1, f_2\}$ is a quasiconvex pair, iff for every pair of nondecreasing functions h_1, h_2, the function $h_1 \circ f_1 + h_2 \circ f_2$ is quasiconvex.

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Remark

The result that the sum of two convex functions is convex, and so quasiconvex, cannot be re-obtained with our results ☹.
But

- We show that our class of functions, for which the sum of quasiconvex functions is quasiconvex, contains not trivial functions and
- ...

Other properties that are preserved under summation in a quasiconvex family

f is semistrictly quasiconvex if

$$[x, y \in \operatorname{dom} f, f(x) \neq f(y)] \Rightarrow \left[f(tx + (1-t)y) < \max\left\{f(x), f(y)\right\}, \forall\, t \in]0,1[.\right]$$

The sum of two semistrictly quasiconvex functions is not necessarily semistrictly quasiconvex.

Example

$f_1\,(x) = x$ and $f_2(x) = \min\left\{-x, -\frac{x}{2}\right\}$.
f_1, f_2 : semistrictly quasiconvex, but their sum is not.

Proposition

Let $f_1, f_2, \ldots, f_m$ be lsc and semistrictly quasiconvex (and so quasiconvex) functions. If $\{f_i :\ i \in J\}$ is a quasiconvex family, then the sum $f_1 + f_2 + \cdots + f_m$ is also semistrictly quasiconvex.

The $\mathcal{C}$-family

$$C \subseteq \mathbb{R}^n$$

$$C^\infty \doteq \{v \in \mathbb{R}^n : \exists\, t_k \to +\infty, \exists\, x_k \in C, \frac{x_k}{t_k} \to v\}.$$ (asymptotic cone).

Definition ([[10], [9]])

It is said that f belongs to $\mathcal{C}$ if for all $x \in \mathrm{dom}\, f$ and all $v \in (\mathrm{dom}\, f)^\infty$, $v \neq 0$, one has either

(i) $0 \leq t \mapsto f(x + tv)$ is nonincreasing, or

(ii) $\lim_{t \to +\infty} f(x + tv) = +\infty$.

Remark: f convex or coercive then $f \in \mathcal{C}$;

Proposition

Let $f_1, f_2, \dots, f_m$ be lsc functions on X, such that

$$\{f_i : i \in J\} \text{ is a quasiconvex family.}$$

If $f_i \in \mathcal{C}$ for all $i \in J$, then $f_1 + f_2 + \cdots + f_m \in \mathcal{C}$.

Q-subdifferential ([14])

$f : \mathbb{R}^n \to \mathbb{R}$,
$\partial^Q f(x) = \left\{(v,t) \in \mathbb{R}^{n+1} : \langle v, x\rangle \geq t \text{ and } f(y) \geq f(x) \text{ if } \langle v, y\rangle \geq t\right\}$.

Theorem ([14])

Let $f, g : \mathbb{R}^n \to \mathbb{R}$ For each $x \in \mathbb{R}^n$, assume that at least one of the following conditions is satisfied:

(i) $[f < f(x)] \subseteq [g < g(x)]$

(ii) $[g < g(x)] \subseteq [f < f(x)]$

(iii) $\partial^Q f(x) \subseteq \partial^Q g(x)$

(iv) $\partial^Q g(x) \subseteq \partial^Q f(x)$.

Then $f + g$ is quasiconvex.

Proposition

Let f, g be as in Theorem 8. Then $\{f, g\}$ is a quasiconvex pair, and so $f + g$ and $\min\{f, g\}$ are quasiconvex.

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