# The Boosted Difference of Convex functions Algorithm

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School of Mathematical Sciences
University of Southampton, UK

joint work with **Francisco J. Aragón Artacho**

**Variational Analysis and Optimisation Webinar**

**9th June 2021**

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# Outline

1. Introduction
2. The Boosted DC Algorithm
   - The algorithm: Adding a line search step to DCA
   - Convergence under the Kurdyka–Łojasiewicz property
3. Numerical experiments
   - A DC problem in biochemistry
   - The Minimum Sum-of-Squares Clustering Problem
   - The Multidimensional Scaling Problem
4. References

# A nonconvex optimization problem

We will focus on the nonconvex optimization problem

$$(\mathcal{P})\ \underset{x\in\mathbb{R}^m}{\text{minimize}}\ \ g(x)-h(x)=:\phi(x),$$

where $g,h:\mathbb{R}^m\to\mathbb{R}\cup\{+\infty\}$ are convex functions with

$$\inf_{x\in\mathbb{R}^m}\phi(x)>-\infty.$$

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The following assumptions are made:

- WLOG $g$ and $h$ are strongly convex with modulus $\rho > 0$ (otherwise, take $\widetilde{g}(x) := g(x) + \frac{\rho}{2}\|x\|^2$ and $\widetilde{h}(x) := h(x) + \frac{\rho}{2}\|x\|^2$).

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- $g$ is continuously differentiable on an open set containing $\operatorname{dom} h$.
- $h$ is subdifferentiable at every point in $\operatorname{dom} h$; i.e., $\partial h(x) \neq \emptyset$ for all $x \in \operatorname{dom} h$.

# First-order necessary optimality condition and critical points

$$(\mathcal{P}) \ \underset{x\in\mathbb{R}^m}{\text{minimize}} \ \ \phi(x) := g(x) - h(x), \text{ with } g \text{ smooth and } h \text{ convex}$$

**Fact (First-order necessary optimality condition)**

*If $x^* \in dom\,\phi$ is an optimal solution of $(\mathcal{P}) \Rightarrow \partial h(x^*) = \{\nabla g(x^*)\}$.*

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We say that $\bar{x}$ is a critical point of $(\mathcal{P})$ if $\nabla g(\bar{x}) \in \partial h(\bar{x})$.

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**Example**

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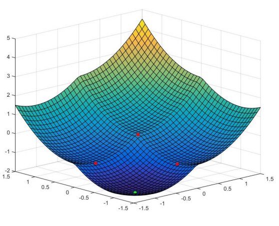
Then, $\phi$ has $2^m$ critical points (any $x \in \{-1, 0\}^m$), and only one point $x^* := (-1, \ldots, -1)$ satisfying $\partial h(x^*) = \{\nabla g(x^*)\}$, which is the global minimum of $\phi$.



# Previous works: linearizing the nonconvex part

## Fukushima–Mine'81

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- In 1981 Fukushima and Mine introduced two algorithms to minimize a composite function $g - h$, where $g$ is (strictly) convex (*possibly nonsmooth*) and $h$ is smooth (*possibly nonconvex*).

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**ALGORITHM 1 (FM'81):** Fix some parameters $\alpha > 0$ and $0 < \beta < 1$. Let $x_0$ be any initial point and set $k := 0$.

1. Find the solution $y_k$ of

$$(\mathcal{P}_k)\ \underset{y\in\mathbb{R}^m}{\text{minimize}}\ \ g(y) - \langle \nabla h(x_k), y\rangle.$$

and set $d_k := y_k - x_k$. If $d_k = 0 \Rightarrow$ **stop** and **return** $x_k$.

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3. Set $x_{k+1} := x_k + \lambda_k d_k$, $k := k + 1$ and **go to** Step 1.

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## Le Thi–Pham Dinh–El Bernoussi'86: DC algorithm

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**ALGORITHM 2 (DCA):** Let $x_0$ be any initial point and set $k := 0$.

1. Choose $u_k \in \partial h(x_k)$
2. Find a solution $y_k$ of

$$(\mathcal{P}_k) \ \underset{y\in\mathbb{R}^m}{\text{minimize}} \ \ g(y) - \langle y, u_k\rangle.$$

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# FM'81 and DCA when $h$ is smooth?

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**Proposition**

*If $g$ and $h$ are strongly convex with constant $\rho > 0$, then*
$$\phi(y_k) \leq \phi(x_k) - \rho\|d_k\|^2 \quad \forall k \in \mathbb{N}.$$

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$$\phi(y_k) \le \phi(x_k) - \rho\|d_k\|^2 \quad \forall k \in \mathbb{N}.$$

$\Rightarrow$ If $0 < \alpha \le \rho$, the iterations of FM'81 and DCA coincide.

# FM'81 and DCA

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$$u_k \in \partial h(x_k), y_k \in \partial g^*(u_k)$$

- FM'81 is based on the fact that $d_k := y_k - x_k$ is a descent direction at $x_k$: it holds $\phi'(x_k; d_k) \leq -\rho\|d_k\|^2$.
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Advantages: Simplicity, works well in practice, does not require any line search.

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Drawbacks: It can be very slow. Can it be accelerated?
Yes, if $g$ is smooth, thanks to the fact that

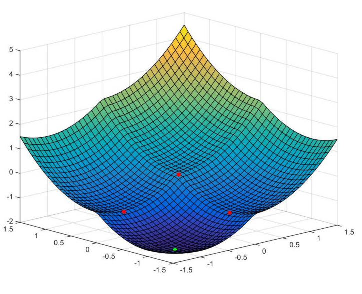
$$\boxed{\phi'(y_k; d_k) \leq -\rho\|d_k\|^2.}$$

# DCA can be slow and get easily trapped by critical points

## Example (Revisited)

Consider the DC function $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined as $\phi := g - h$, where

$$g(x) = \frac{3}{2}\left(x_1^2 + x_2^2\right) + x_1 + x_2 \quad \text{and} \quad h(x) = |x_1| + |x_2| + \frac{1}{2}\left(x_1^2 + x_2^2\right).$$



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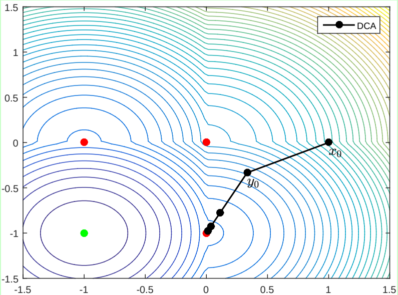

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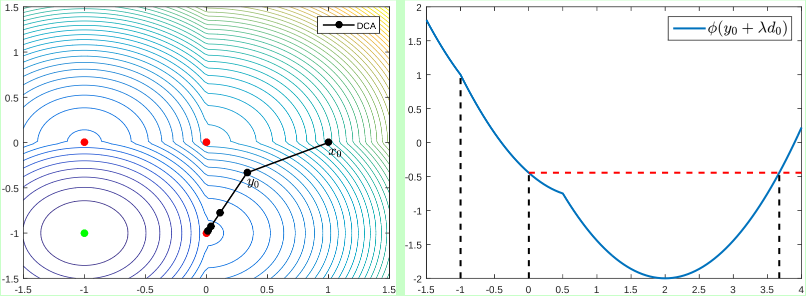
Consider the DC function $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined as $\phi := g - h$, where

$$g(x) = \frac{3}{2}\left(x_1^2 + x_2^2\right) + x_1 + x_2 \quad \text{and} \quad h(x) = |x_1| + |x_2| + \frac{1}{2}\left(x_1^2 + x_2^2\right).$$



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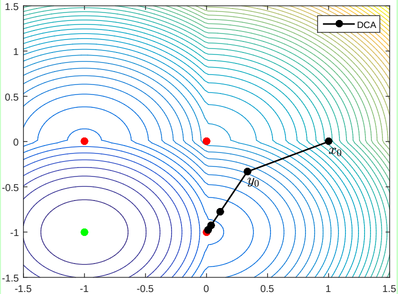


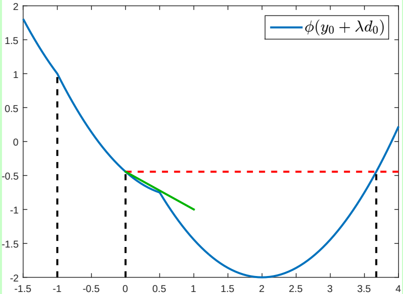


$$\boxed{\phi'(y_0; d_0) \leq -\rho \|d_0\|^2}$$

# Outline


1 Introduction

2 The Boosted DC Algorithm
- The algorithm: Adding a line search step to DCA
- Convergence under the Kurdyka–Łojasiewicz property

3 Numerical experiments
- A DC problem in biochemistry
- The Minimum Sum-of-Squares Clustering Problem
- The Multidimensional Scaling Problem

4 References

# The Boosted DC Algorithm

$$\phi(x) := g(x) - h(x), u_k \in \partial h(x_k), y_k \in \partial g^*(u_k)$$

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Hence

$$\phi'(y_k; d_k) \leq \langle \nabla g(y_k) - v, d_k \rangle = \langle u_k - v, y_k - x_k \rangle \leq -\rho\|d_k\|^2. \quad \blacksquare$$

# The Boosted DC Algorithm

**BDCA** (*Boosted DC Algorithm*)

Fix $\alpha > 0$ and $0 < \beta < 1$. Let $x_0$ be any initial point and set $k := 0$.

1. Select $u_k \in \partial h(x_k)$ and find the unique solution $y_k$ of the problem

$$(\mathcal{P}_k) \ \underset{x \in \mathbb{R}^m}{\text{minimize}} \ g(x) - \langle u_k, x \rangle.$$

2. Set $d_k := y_k - x_k$. If $d_k = 0 \Rightarrow$ **stop** and **return** $x_k$.
3. Choose any $\overline{\lambda}_k \geq 0$. Set $\lambda_k := \overline{\lambda}_k$.
   **while** $\phi(y_k + \lambda_k d_k) > \phi(y_k) - \alpha\lambda_k^2 \|d_k\|^2$ **do** $\lambda_k := \beta\lambda_k$.
4. Set $x_{k+1} := y_k + \lambda_k d_k$, $k := k + 1$, and **go to** Step 1.

# DCA vs BDCA

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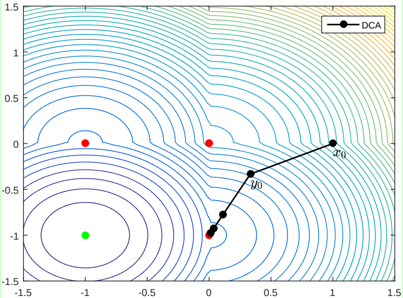


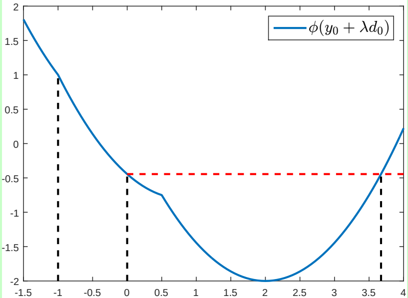

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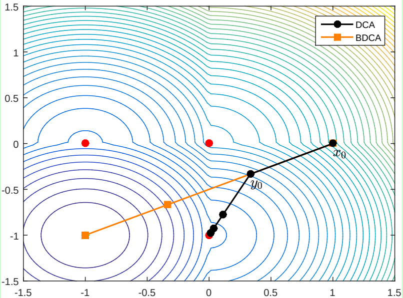


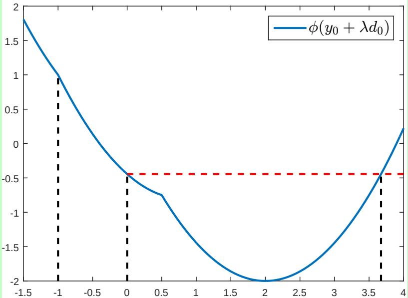

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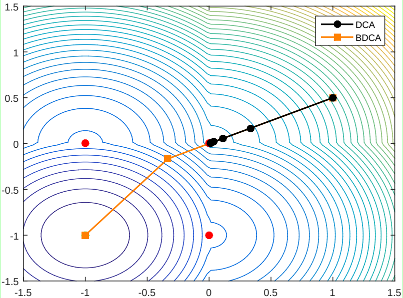


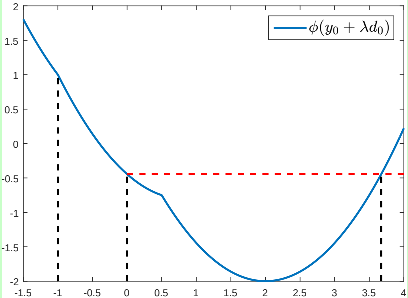

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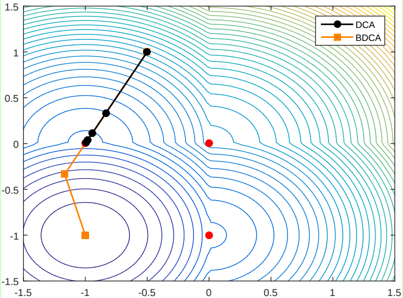


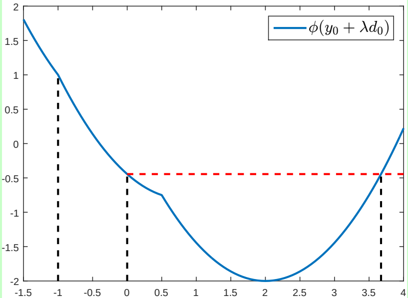

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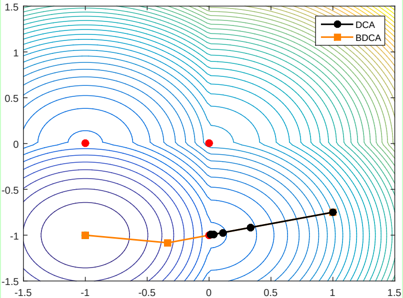


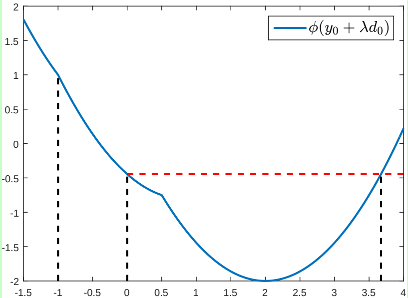

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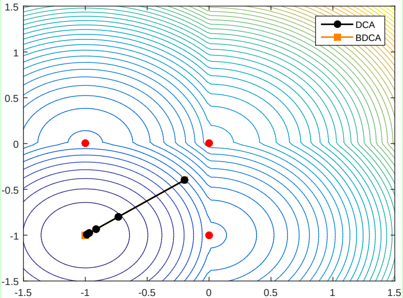


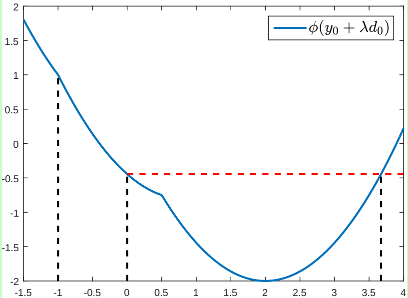

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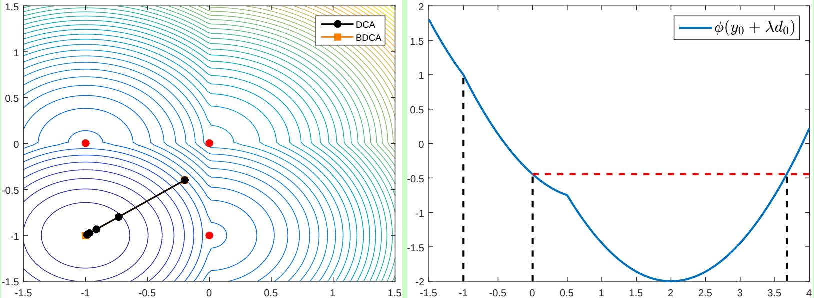



| | $(-1,-1)$ | $(-1,0)$ | $(0,-1)$ | $(0,0)$ |
|---|---|---|---|---|
| DCA | 249,763 | 249,841 | 250,204 | 250,192 |
| BDCA | 996,104 | 1,922 | 1,974 | 0 |

Table: For one million random starting points in $[-1.5, 1.5]^2$, we count the sequences converging to each of the four stationary points.

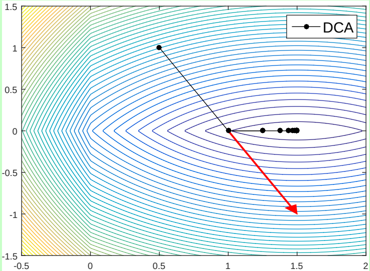
# Why to restrict to the case where $g$ is differentiable?

## Example (Failure of BDCA when $g$ is not differentiable)

Consider the following modification of the previous example

$$g(x) = -\frac{5}{2}x_1 + x_1^2 + x_2^2 + |x_1| + |x_1| \quad \text{and} \quad h(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right),$$

so $h$ is differentiable but $g$ is not. Let $x_0 = (0.5, 1)$. The point generated by DCA is $y_0 = (1, 0)$ and $d_0 = y_0 - x_0 = (0.5, -1)$ is not a descent direction for $\phi$ at $y_0$:



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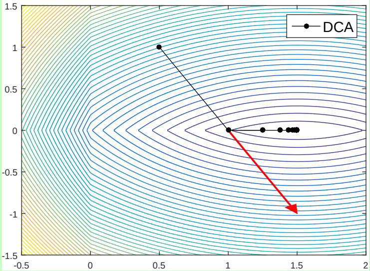


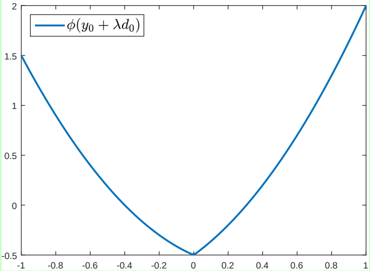

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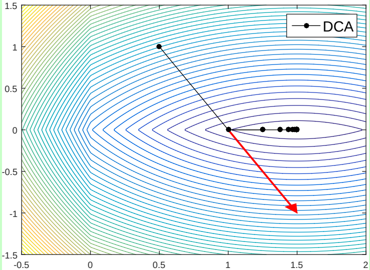


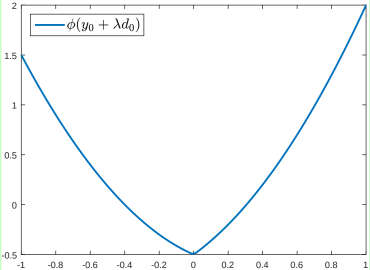


## Proposition

*Let $\phi = g - h$, where $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are convex and $h$ is differentiable. If $0 \notin \partial_C \phi(y_k)$, then $\phi'(y_k; y_k - x_k) < 0$.*

# Convergence of BDCA

$$(\mathcal{P})\ \underset{x\in\mathbb{R}^m}{\text{minimize}}\ \ \phi(x) := g(x) - h(x)$$

Our convergence results follow the ideas from

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## Theorem

For any $x_0 \in \mathbb{R}^m$, either BDCA returns a critical point of $(\mathcal{P})$ or it generates an infinite sequence such that the following holds.

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3. $\sum_{k=0}^{+\infty} \|d_k\|^2 < +\infty$. Further, if there is some $\overline{\lambda}$ such that $\lambda_k \leq \overline{\lambda}$ for all $k$, then $\sum_{k=0}^{+\infty} \|x_{k+1} - x_k\|^2 < +\infty$.

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# The Kurdyka–Łojasiewicz property

## Definition

Let $f : \mathbb{R}^m \to \mathbb{R}$ be a locally Lipschitz function. We say that $f$ satisfies the strong Kurdyka–Łojasiewicz inequality at $x^* \in \mathbb{R}^m$ if there exist $\eta \in ]0, +\infty[$, a neighborhood $U$ of $x^*$, and a concave function $\varphi : [0, \eta] \to [0, +\infty[$ such that:

1. $\varphi(0) = 0$;
2. $\varphi$ is of class $\mathcal{C}^1$ on $]0, \eta[$;
3. $\varphi' > 0$ on $]0, \eta[$;
4. for all $x \in U$ with $f(x^*) < f(x) < f(x^*) + \eta$ we have

$$\varphi'(f(x) - f(x^*))\,\mathrm{dist}\,(0, \partial_C f(x)) \geq 1.$$

Here $\partial_C f$ stands for the Clarke subdifferential

$$\partial_C f(\bar{x}) = \mathrm{co}\left\{ \lim_{x \to \bar{x},\, x \notin \Omega_f} \nabla f(x) \right\},$$

where $\mathrm{co}$ stands for the convex hull and $\Omega_f$ denotes the set of Lebesgue measure zero where $f$ fails to be differentiable.

# Convergence under the Kurdyka–Łojasiewicz property

$$(\mathcal{P}) \ \underset{x\in\mathbb{R}^m}{\text{minimize}} \ \ \phi(x) := g(x) - h(x)$$

**Theorem (Convergence)**

Let $\{x_k\}$ be the sequence generated by the BDCA. Suppose that $\{x_k\}$ has a cluster point $x^*$, that $\nabla g$ is locally Lipschitz around $x^*$ and that $\phi$ satisfies the strong Kurdyka–Łojasiewicz inequality at $x^*$.
Then $\{x_k\}$ converges to $x^*$, which is a critical point of $(\mathcal{P})$.

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## Proof

- Technical but “standard”.
- $\lambda_k$ can be zero or unbounded!
- We either need Clarke’s subdifferential or to assume that $-\phi$ satisfies the Kurdyka–Łojasiewicz inequality:

$$\nabla g(y_k) - \nabla g(x_k) \in \partial h(x_k) - \nabla g(x_k) = \boxed{\partial_C\left(-\phi(x_k)\right) = -\partial_C \phi(x_k)}$$

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## Theorem (Rate)

Suppose that the sequence $\{x_k\}$ generated by the BDCA has the limit point $x^*$, that $\nabla g$ is locally Lipschitz continuous around $x^*$ and $\phi$ satisfies the strong Kurdyka–Łojasiewicz inequality at $x^*$ with $\varphi(t) = Mt^{1-\theta}$ for some $M > 0$ and $0 \leq \theta < 1$. Then:

1. if $\theta = 0$, then $\{x_k\}$ converges in a finite number of steps to $x^*$;

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$$(\mathcal{P}) \ \underset{x\in\mathbb{R}^m}{\text{minimize}} \ \ \phi(x) := g(x) - h(x)$$

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1. if $\theta = 0$, then $\{x_k\}$ converges in a finite number of steps to $x^*$;
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3. if $\theta \in \left]\frac{1}{2}, 1\right[$, then $\exists \eta > 0$ s.t. $\|x_k - x^*\| \leq \eta k^{-\frac{1-\theta}{2\theta-1}}$ for all large $k$.

# How to choose the trial step size $\overline{\lambda}_k$?

**BDCA** (*Boosted DC Algorithm*)

Fix $\alpha > 0$ and $0 < \beta < 1$. Let $x_0$ be any initial point and set $k := 0$.

1. Select $u_k \in \partial h(x_k)$ and $y_k \in \partial g^*(u_k)$.
2. Set $d_k := y_k - x_k$. If $d_k = 0 \Rightarrow$ **stop** and **return** $x_k$.
3. Choose any $\overline{\lambda}_k \geq 0$. Set $\lambda_k := \overline{\lambda}_k$.
   **while** $\phi(y_k + \lambda_k d_k) > \phi(y_k) - \alpha\lambda_k^2\|d_k\|^2$ **do** $\lambda_k := \beta\lambda_k$.
4. Set $x_{k+1} := y_k + \lambda_k d_k, k := k + 1$, and **go to** Step 1.

One possibility would be to set $\overline{\lambda}_k = \overline{\lambda}$ for all $k$.

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One possibility would be to set $\overline{\lambda}_k = \overline{\lambda}$ for all $k$. Instead, we propose:

**Self-adaptive trial step size**

Fix $\gamma > 1$. Set $\overline{\lambda}_0 := 0$. Choose some $\overline{\lambda}_1 > 0$ and obtain $\lambda_1$ by BDCA.
For any $k \geq 2$:

1. **if** $\lambda_{k-2} = \overline{\lambda}_{k-2}$ **and** $\lambda_{k-1} = \overline{\lambda}_{k-1}$ **then** set $\overline{\lambda}_k := \gamma\lambda_{k-1}$;
   **else** set $\overline{\lambda}_k := \lambda_{k-1}$.
2. Obtain $\lambda_k$ from $\overline{\lambda}_k$ by the backtracking step of BDCA.

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# **Experiment 1:** Finding steady states of biochemical networks

We were interested in finding a solution to the problem

$$\underset{x\in\mathbb{R}^m}{\text{minimize}}\ \ \phi(x) := \|p(x) - c(x)\|^2$$

where

$$p(x) := [F,R]\exp\left(p + [F,R]^T x\right) \quad \text{and} \quad c(x) := [R,F]\exp\left(p + [F,R]^T x\right),$$

and $F, R \in \mathbb{Z}_{\geq 0}^{m\times n}$ ($m$ molecular species, $n$ reversible elementary reactions).

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- It is not difficult to prove that $f$ is real analytic
  $\Rightarrow f$ satisfies the Łojasiewicz property with some exponent $\theta \in [0, 1)$.
- We can then apply BDCA to the functions

  $$g(x) := 2\left(\|p(x)\|^2 + \|c(x)\|^2\right) + \frac{\rho}{2}\|x\|^2 \quad \text{and} \quad h(x) := \|p(x)+c(x)\|^2 + \frac{\rho}{2}\|x\|^2,$$

  for any $\rho > 0$. Our results guarantee the convergence of the sequence generated by BDCA, as long as the sequence is bounded.

# Finding steady states of biochemical networks

## A boring table of computational results...

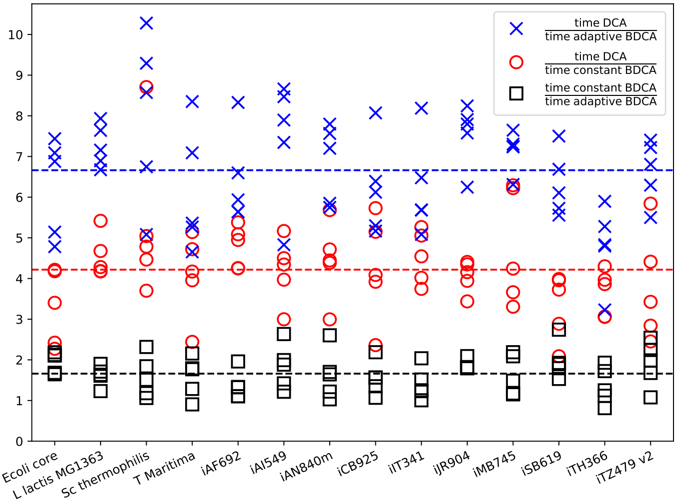
Taking $\rho := 100$, $\overline{\lambda}_k := 50$ (constant), $\beta := 0.5$ and $\alpha := 0.4$, we obtain:

| DATA | | | ALGORITHMS: Time Spent (seconds) | | | | | | RATIO (avg.) DCA/BDCA | |
|---|---|---|---|---|---|---|---|---|---|---|
| Model Name | m | n | BDCA | | | DCA | | | | |
| | | | min. | max. | avg. | min. | max. | avg. | iter. | time |
| Ecoli core | 72 | 94 | 16 | 26 | 20 | 68 | 105 | 87 | 4.9 | 4.4 |
| L lactis MG1363 | 486 | 615 | 2926 | 4029 | 3424 | 14522 | 18212 | 16670 | 5.2 | 4.9 |
| Sc thermophilis | 349 | 444 | 291 | 553 | 358 | 1302 | 2004 | 1611 | 4.9 | 4.5 |
| T Maritima | 434 | 554 | 1333 | 2623 | 1920 | 5476 | 12559 | 8517 | 4.7 | 4.4 |
| iAF692 | 466 | 546 | 1677 | 2275 | 1967 | 8337 | 11187 | 9466 | 5.3 | 4.8 |
| iAI549 | 307 | 355 | 177 | 254 | 209 | 665 | 1078 | 913 | 4.9 | 4.4 |
| iAN840m | 549 | 840 | 3229 | 6939 | 4721 | 16473 | 28957 | 21413 | 5.0 | 4.5 |
| iCB925 | 416 | 584 | 1831 | 2450 | 2133 | 7358 | 11465 | 9887 | 5.0 | 4.6 |
| iIT341 | 425 | 504 | 1925 | 2883 | 2302 | 9434 | 20310 | 12262 | 5.7 | 5.3 |
| iJR904 | 597 | 915 | 6363 | 9836 | 7623 | 24988 | 43640 | 33621 | 4.4 | 4.8 |
| iMB745 | 528 | 652 | 2629 | 5091 | 4252 | 16438 | 25172 | 20269 | 5.0 | 4.8 |
| iSB619 | 462 | 598 | 2407 | 5972 | 3323 | 8346 | 25468 | 13967 | 4.3 | 4.2 |
| iTH366 | 587 | 713 | 3310 | 5707 | 4464 | 13613 | 30044 | 20715 | 5.0 | 4.6 |
| iTZ479 v2 | 435 | 560 | 1211 | 2656 | 2216 | 7368 | 12592 | 10120 | 4.9 | 4.6 |

For each model, we selected a random kinetic parameter $p \in [-1, 1]^{2n}$ and 10 initial random points $x_0 \in [-2, 2]^m$. BDCA was run 1000 iterations, DCA was run until it reached the same value obtained by BDCA.

# Finding steady states of biochemical networks

## Comparison of the constant and self-adaptive trial step size strategy for BDCA



# Applications to Biochemistry

## Nature Protocols (2019)

**The COBRA Toolbox**

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### The COnstraint-Based Reconstruction and Analysis Toolbox

View The COBRA Toolbox **source code** on **GitHub**.

Installation Tutorials Contributing Documentation

The COnstraint-Based Reconstruction and Analysis Toolbox is a MATLAB software suite for quantitative prediction of cellular and multicellular biochemical networks with constraint-based modelling. It implements a comprehensive collection of basic and advanced modelling methods, including reconstruction and model generation as well as biased and unbiased model-driven analysis methods.

It is widely used for modelling, analysing and predicting a variety of metabolic phenotypes using genome-scale biochemical networks.

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# The Minimum Sum-of-Squares Clustering Problem

Clustering is an unsupervised technique for data analysis whose objective is to group a collection of objects into clusters based on similarity.

- Let $A = \{a^1, \ldots, a^n\}$ be a finite set of points in $\mathbb{R}^m$, which represent the data points to be grouped.
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The *Minimum Sum-of-Squares Clustering* criterion, one tries to minimize the Euclidean distance of each data point to the centroid of its clusters, denoted by $X := (x^1, \ldots, x^k) \in \mathbb{R}^{m \times k}$:

$$\underset{X \in \mathbb{R}^{m \times k}}{\text{minimize}} \ \phi(X) := \frac{1}{n} \sum_{i=1}^{n} \min_{j=1,\ldots,k} \|x^j - a^i\|^2$$

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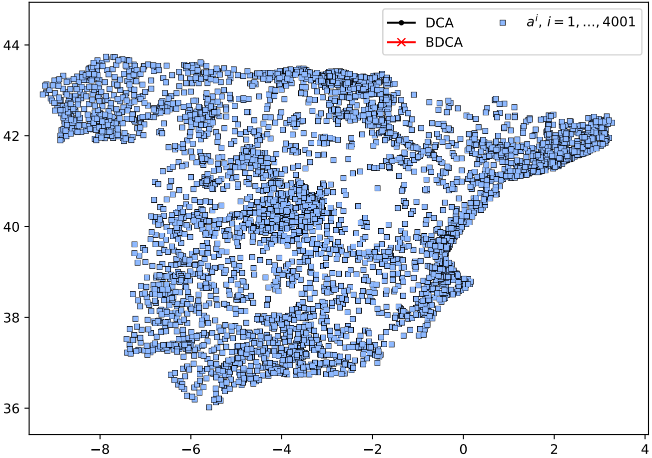
$$h(X) := \frac{1}{n}\sum_{i=1}^{n} \max_{j=1,\ldots,k} \sum_{t=1, t\neq j}^{k} \left\|x^t - a^i\right\|^2 + \frac{\rho}{2}\|X\|^2. \quad \text{(strongly convex but nonsmooth).}$$

# **Experiment 2:** Clustering the Spanish cities in the peninsula

**PROBLEM:** Find a partition into 5 clusters of the 4001 Spanish cities in the peninsula with more than 500 residents[1].

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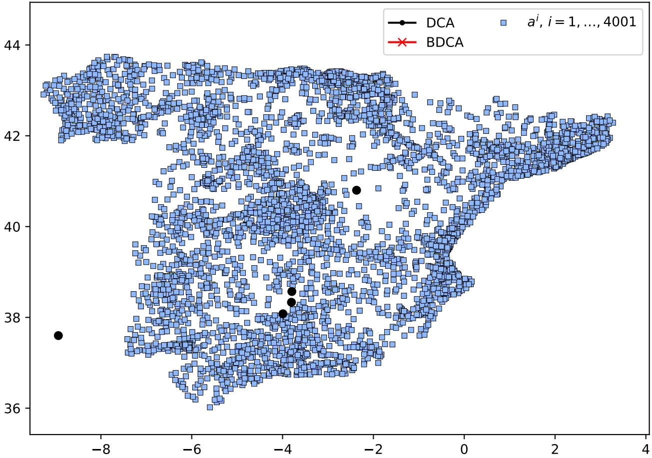


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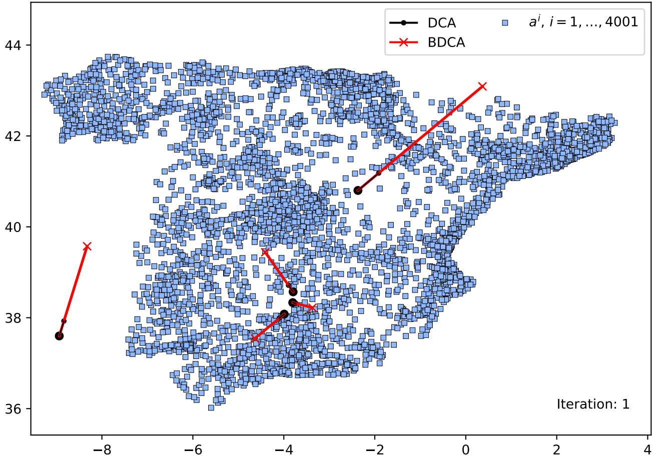


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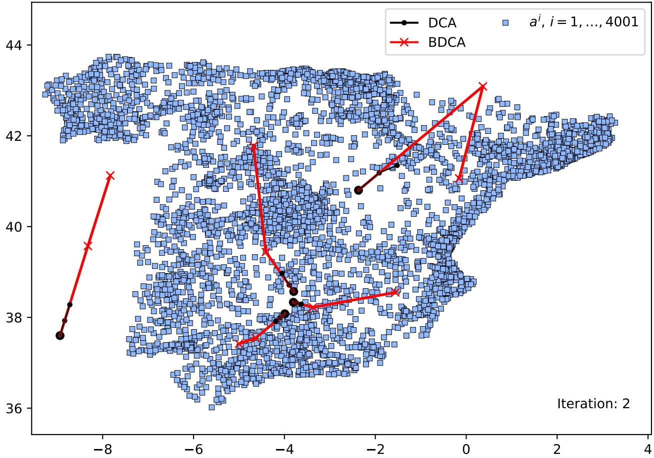


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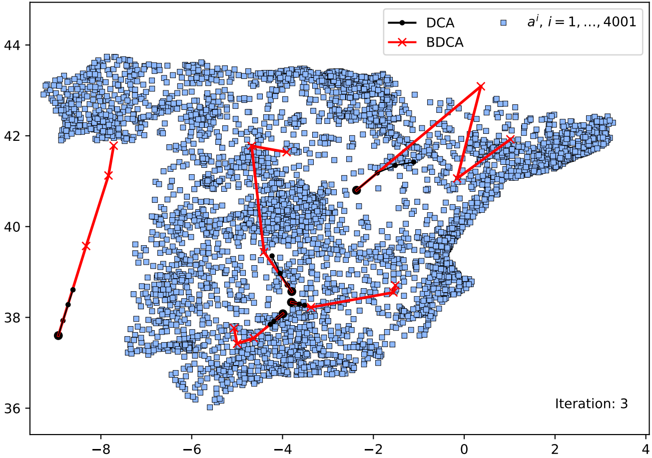


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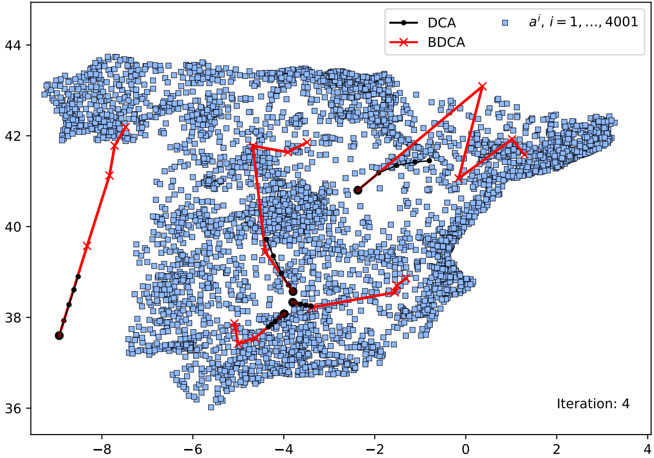


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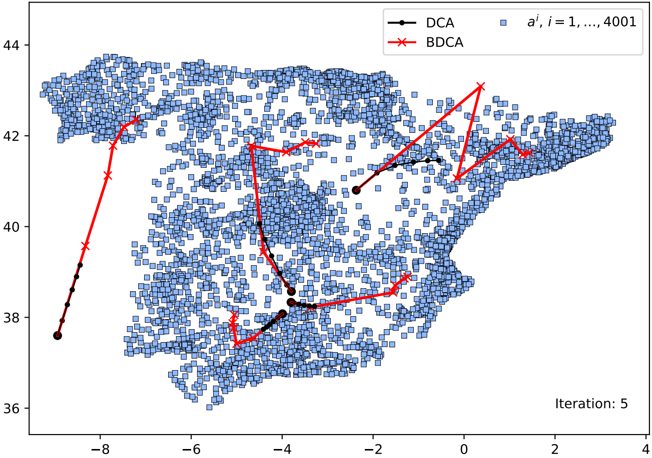


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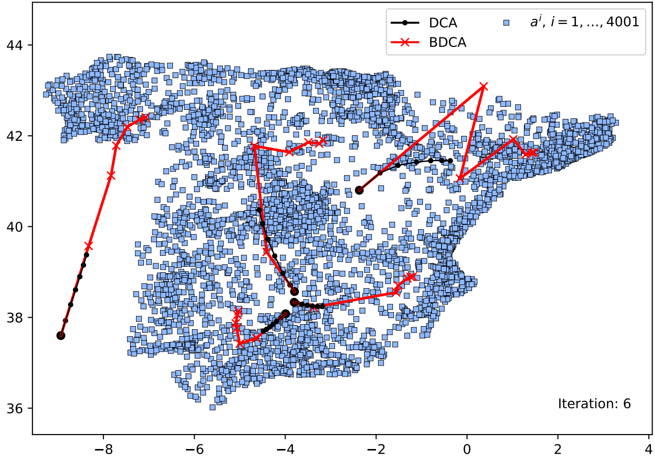


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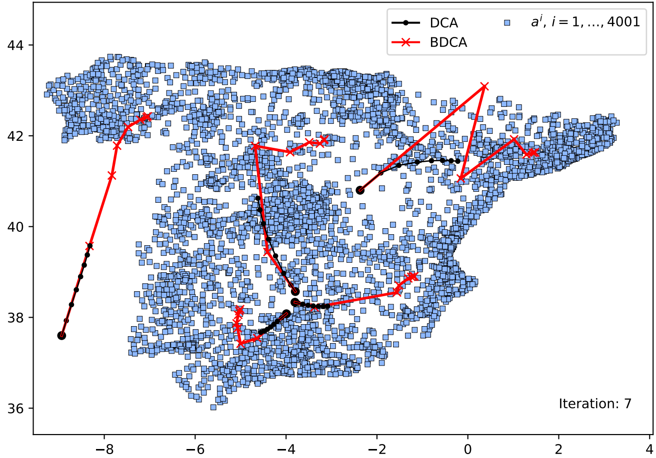


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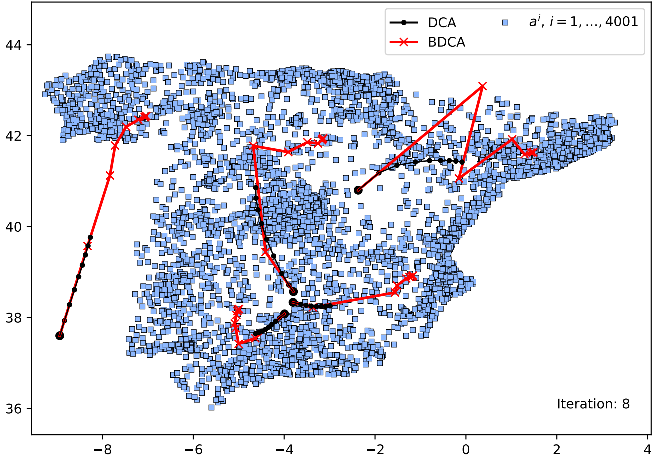


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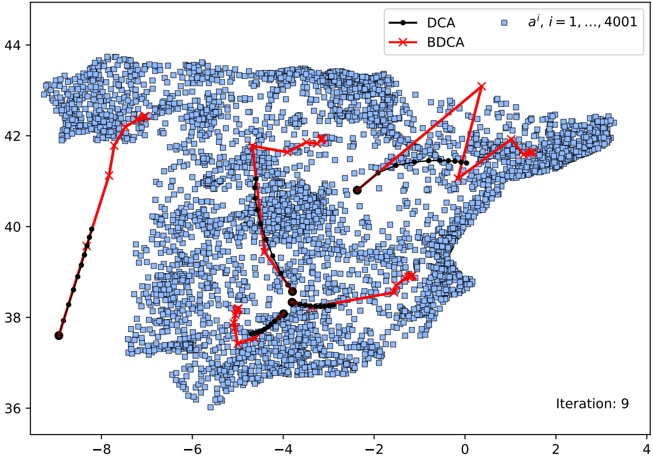


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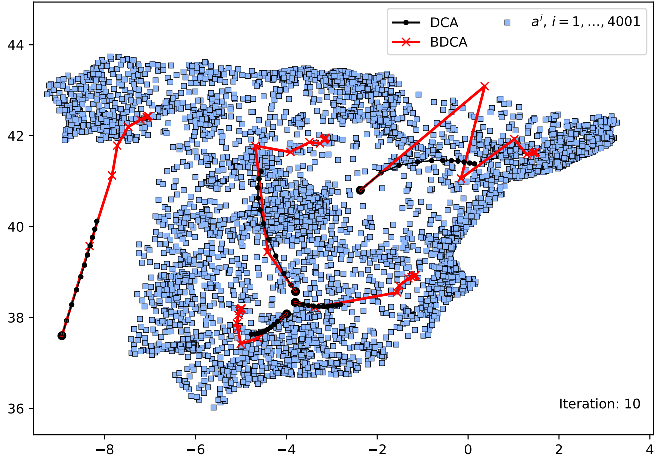


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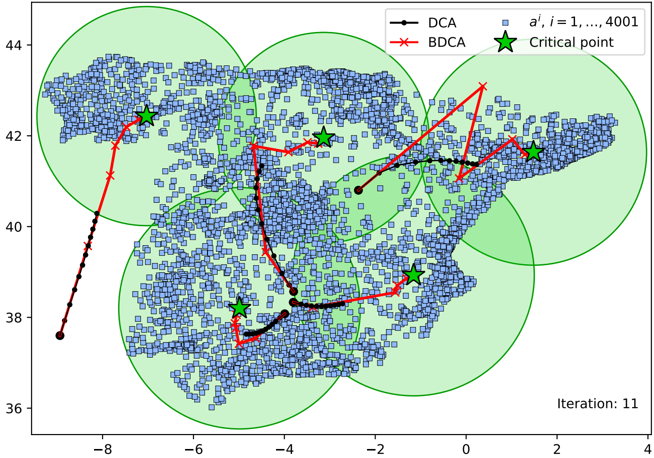


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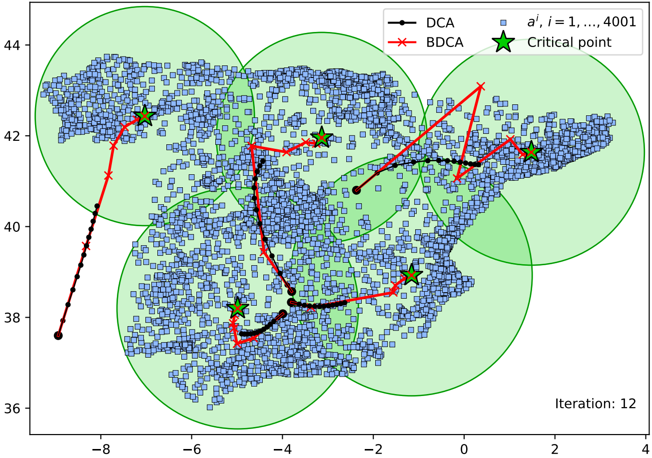


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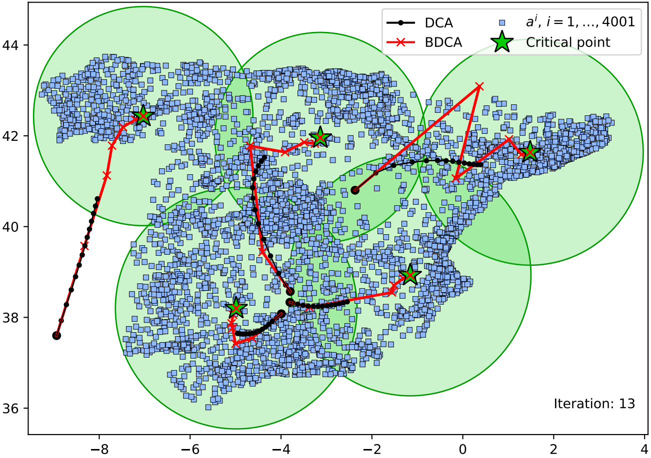


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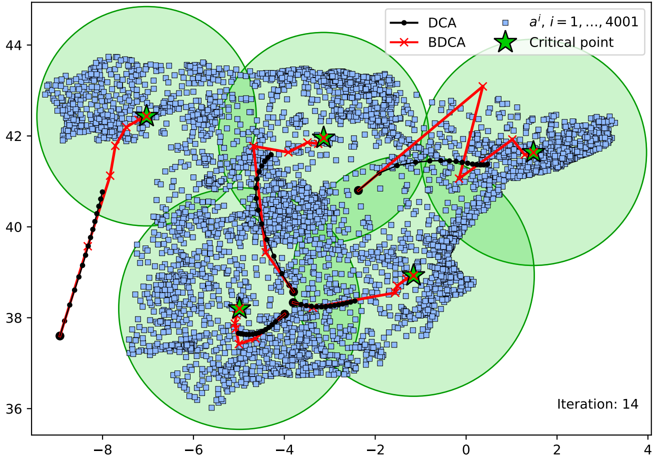


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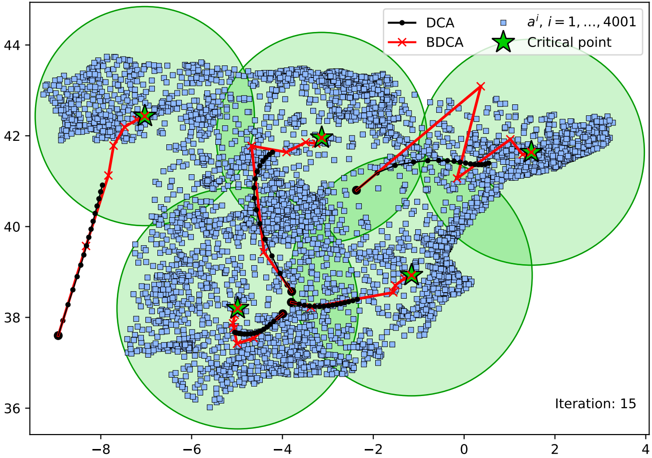


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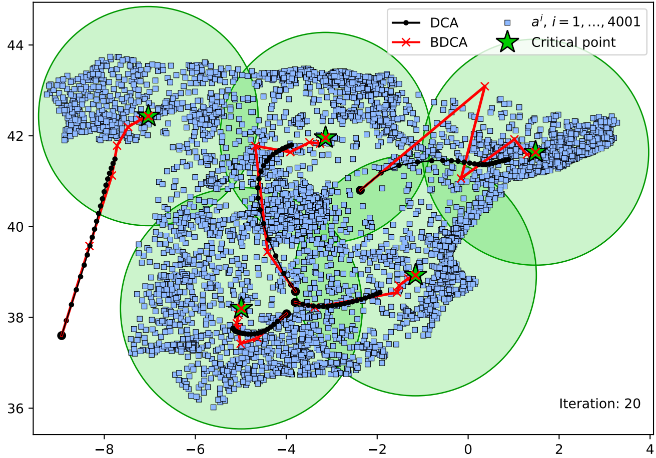


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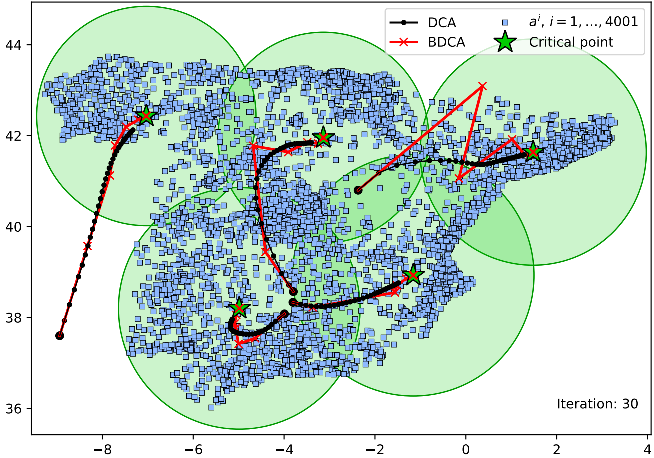


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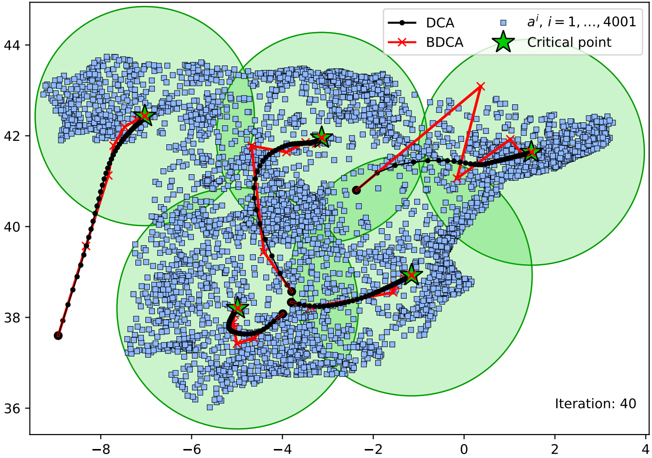


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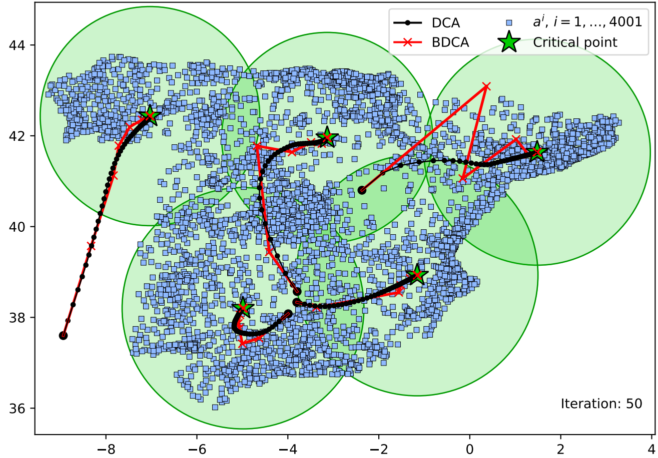


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For 100 random starting points, BDCA was stopped when the relative error of $\phi$ was smaller than $10^{-3}$. Then, DCA was run from the same starting point until the same value of the objective function was reached (which did not happen in 31 instances).

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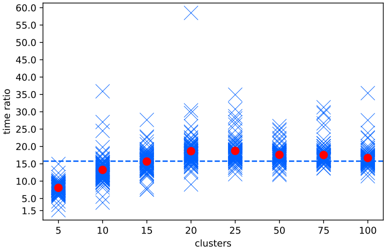


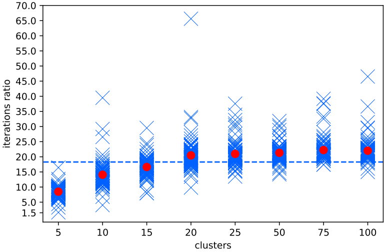


Figure: Comparison between DCA and BDCA. We represent the ratios of running time (left) and number of iterations (right) between DCA and BDCA.

# Experiment 3: Clustering random points in an $m$-dimensional box

We took $n$ random points in $\mathbb{R}^m$ (normal distribution), with $n \in \{500, 1000, 5000, 10000\}$ and $m \in \{2, 5, 10, 20\}$. For each $(n, m)$, 10 random starting points were chosen. Then:

- BDCA was run to solve the $k$-clustering, with $k \in \{5, 10, 15, 20, 25, 50, 75, 100\}$.
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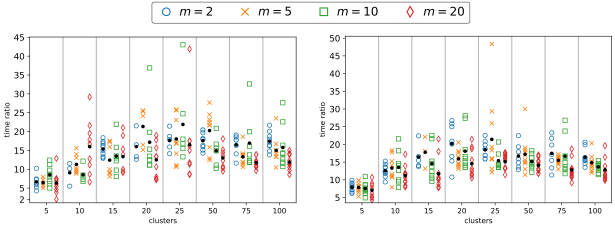
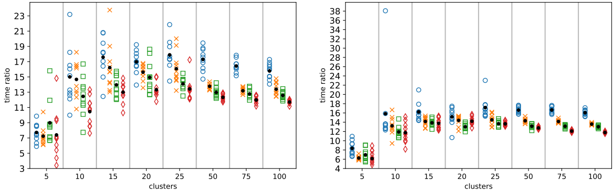



$n = 500$ $n = 1000$

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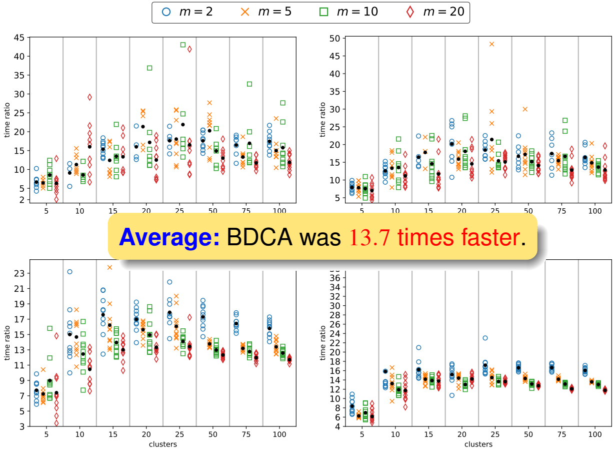


**Average:** BDCA was 13.7 times faster.

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# Outline

1 Introduction

2 The Boosted DC Algorithm
- The algorithm: Adding a line search step to DCA
- Convergence under the Kurdyka–Łojasiewicz property

3 Numerical experiments
- A DC problem in biochemistry
- The Minimum Sum-of-Squares Clustering Problem
- The Multidimensional Scaling Problem

4 References

# The Multidimensional Scaling Problem

Suppose that we have a table of distances between some objects, known as the dissimilarity matrix. If the objects are $n$ points $x^1, x^2, \ldots, x^n$ in $\mathbb{R}^q$, the dissimilarity matrix can be defined by the Euclidean distance between them:

$$\delta_{ij} = \|x^i - x^j\| := \mathsf{d}_{ij}(X),$$

where we denote by $X$ the $n \times q$ matrix whose rows are $x^1, x^2, \ldots, x^n$.

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Given a target dimension $p \leq q$, the MDS problem consists in finding $n$ points in $\mathbb{R}^p$, which are represented by an $n \times p$ matrix $X^*$, such that

$$Stress(X^*) := \sum_{i<j} w_{ij} \left(\mathsf{d}_{ij}(X^*) - \delta_{ij}\right)^2$$

is smallest, where $w_{ij} \geq 0$ are weights ($w_{ij} = 0$ if $\delta_{ij}$ is missing).

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This can be equivalently formulated as a DC problem by setting

$$g(X) := \frac{1}{2} \sum_{i<j} w_{ij} \mathsf{d}_{ij}^2(X) + \frac{\rho}{2} \|X\|^2,$$

$$h(X) := \sum_{i<j} w_{ij} \delta_{ij} \mathsf{d}_{ij}(X) + \frac{\rho}{2} \|X\|^2,$$

for some $\rho \geq 0$. Moreover, it is clear that $g$ is differentiable while $h$ is not, but $\partial h$ can be explicitly computed.

# Experiment 4: MDS for Spanish cities

Consider the dissimilarity matrix defined by the distances between the 4155 Spanish cities with more than 500 residents, including this time those outside the peninsula. $\Rightarrow$ The optimal value of this MDS problem is zero.

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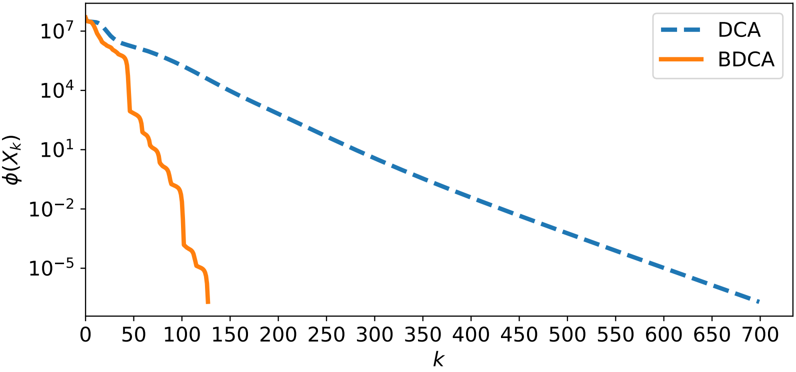

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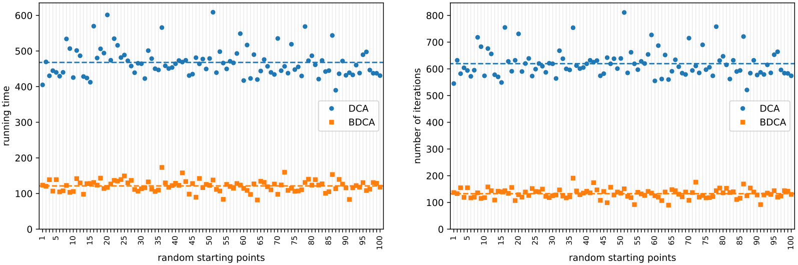


Figure: Running time (left) and number of iterations (right) of DCA and BDCA for 100 random instances.

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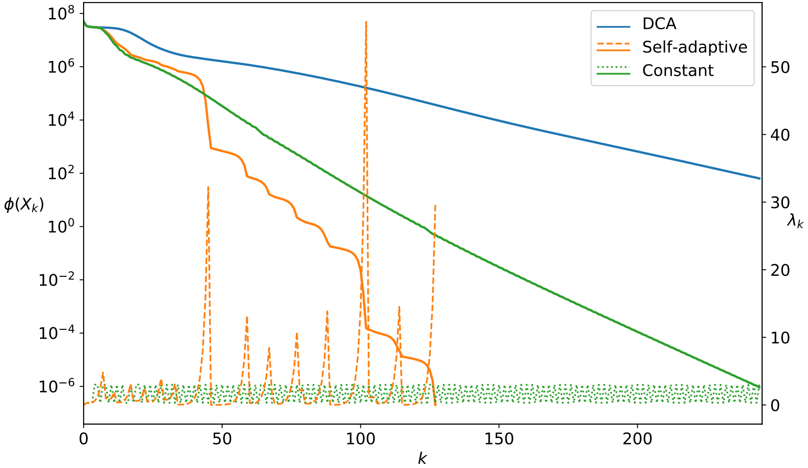

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