Error bounds, amenable cones and beyond

Bruno F. Lourenço
The Institute of Statistical Mathematics

June 16th, 2021
VAO Webinar

Conic linear programming

$$\begin{aligned} \text{minimize} \quad & \langle c, x \rangle \\ \text{subject to} \quad & x \in (\mathcal{L} + a) \cap \mathcal{K} \end{aligned}$$

- $\mathcal{K}$: closed convex cone contained in some space $\mathcal{E}$.
- $\mathcal{L}$: subspace contained in $\mathcal{E}$.
- $a, c \in \mathcal{E}$.

General philosophy: isolate the nonlinearity of the problem into the conic constraints.

- Many good solvers: SeDuMi, SDPT3, SDPA, MOSEK and others.
- Many applications.
  - *Lectures on Modern Convex Optimization* (Ben-Tal and Nemirovski)
  - *MOSEK Modeling Cookbook.* `https://docs.mosek.com/modeling-cookbook`

Feasibility problems over convex cones

Consider the following *feasibility problem over a convex cone* $\mathcal{K}$.

$$\begin{array}{rl} \text{find} & x \\ \text{subject to} & x \in (\mathcal{L} + a) \cap \mathcal{K} \end{array}$$

- $\mathcal{K}$: closed convex cone contained in some space $\mathcal{E}$.
- $\mathcal{L}$: subspace contained in $\mathcal{E}$.
- $a \in \mathcal{E}$.

($\mathcal{L} + a$ is an affine space)

L.
Amenable cones: error bounds without constraint qualifications.
Mathematical Programming **186**, 1–48 (2021)

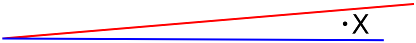
Motivation

Let $\|\cdot\|$ be the Euclidean norm and fix $x \in \mathcal{E}$.

$$\operatorname{dist}(x, \mathcal{L} + a) = \inf\{\|x - y\| \mid y \in \mathcal{L} + a\}$$
$$\operatorname{dist}(x, \mathcal{K}) = \inf\{\|x - y\| \mid y \in \mathcal{K}\}$$
$$\operatorname{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}) = \inf\{\|x - y\| \mid y \in (\mathcal{L} + a) \cap \mathcal{K}\}$$

Fundamental question

Can we estimate $\operatorname{dist}(x, (\mathcal{L} + a) \cap \mathcal{K})$ using $\operatorname{dist}(x, \mathcal{L} + a)$ and $\operatorname{dist}(x, \mathcal{K})$?



- Convergence analysis often leads to this type of questions.

Hölderian error bounds

C_1, C_2: closed convex sets.
$C := C_1 \cap C_2$

Definition (Hölderian error bound)

C_1, C_2 satisfy a **Hölderian error bound** $\overset{\text{def}}{\iff}$ for every bounded set B there exist $\theta_B > 0$, $\gamma_B \in (0,1]$ such that

$$\operatorname{dist}(x,\ C) \leq \theta_B \max_{1 \leq i \leq 2} \operatorname{dist}(x,\ C_i)^{\gamma_B} \quad \forall\ x \in B.$$

If $\gamma_B = \gamma \in (0,1]$ for all B, the bound is **uniform**. If the bound is uniform with $\gamma = 1$, we call it a **Lipschitzian** error bound.

Some known results

Some results: (C_1, C_2: convex sets, with $C_1 \cap C_2 \neq \emptyset$)

- $\mathrm{ri}\, C_1 \cap \mathrm{ri}\, C_2 \neq \emptyset \Rightarrow$ Lipschitzian
- C_1, C_2 are polyhedral $\Rightarrow$ Lipschitzian (Hoffman's Lemma)
- C_1, C_2: basic convex semialgebraic sets $\Rightarrow$ Uniform Hölderian (Borwein, Li and Yao's error bound)
- C_1: affine space, C_2: PSD cone $\Rightarrow$ Uniform Hölderian (Sturm's error bound)

Key issue

Determining **whether** a Hölderian error bound holds. If yes, determing the **exponent** as tightly as possible.

Sturm's bound

S^n: $n \times n$ symmetric matrices.
$\mathcal{S}^n_+$: $n \times n$ positive semidefinite matrices.

Theorem (Sturm's Error Bound)

Suppose $(\mathcal{L} + a) \cap \mathcal{S}^n_+ \neq \emptyset$. *There exists* $\gamma \geq 0$ *such that for every* $\rho > 0$, *there exists* $\kappa_\rho > 0$ *such that*

$$\operatorname{dist}(x, \mathcal{L} + a) \leq \epsilon, \quad \operatorname{dist}(x, \mathcal{S}^n_+) \leq \epsilon, \quad \|x\| \leq \rho$$

implies

$$\operatorname{dist}(x, (\mathcal{L} + a) \cap \mathcal{S}^n_+) \leq \kappa_\rho \epsilon^{(2^{-\gamma})},$$

where $\gamma \leq \min\{n - 1, \dim(\mathcal{L}^\perp \cap \{a\}^\perp), \dim \operatorname{span}(\mathcal{L} + a)\}$.

- Tight!
- γ is connected to the so-called **singularity degree**.

J. F. Sturm.
Error bounds for linear matrix inequalities.
SIAM Journal on Optimization, 10(4):1228–1248, Jan. 2000.

Beyond Sturm's bound

Question

For which cones does a result similar to Sturm's bound holds?

Answer: For **symmetric cones**, a result almost *identical* to Sturm's bound holds. For a new class of cones called **amenable cones**, similar results holds. Three ingredients are needed:

- Amenable cones
- Facial Residual Functions (FRFs)
- Facial Reduction.

Review of faces

- $\mathcal{K}$: closed convex cone
- $\mathcal{F} \subseteq \mathcal{K}$: closed convex cone

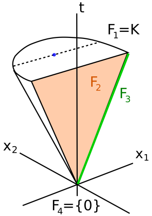
Definition (Face of a cone)

$\mathcal{F}$ is a face of $\mathcal{K}$ $\overset{\text{def}}{\iff}$ if $x + y \in \mathcal{F}$, with $x, y \in \mathcal{K}$, then $x, y \in \mathcal{F}$.

Definition (Exposed face)

$\mathcal{F}$ is an **exposed** face of $\mathcal{K}$ $\overset{\text{def}}{\iff}$ $\mathcal{F} = \mathcal{K} \cap \{z\}^\perp$, for some $z \in \mathcal{K}^*$.

Fact: $\mathcal{F} = \mathcal{K} \cap \operatorname{span} \mathcal{F}$



Amenable cones

Definition (Amenable cones)

$\mathcal{K}$ is **amenable** if for every face $\mathcal{F}$ of $\mathcal{K}$ there is $\kappa > 0$ such that

$$\mathrm{dist}\,(x, \mathcal{F}) \leq \kappa \mathrm{dist}\,(x, \mathcal{K}), \qquad \forall x \in \mathrm{span}\, \mathcal{F}.$$

- Symmetric cones (e.g., PSD cone) are amenable ($\kappa = 1$)
- Polyhedral cones are amenable
- Strictly convex cones are amenable. (p-cones, second order cones and so on)
- Amenability is preserved under linear isomorphism and direct products

Reminders:

$\mathcal{K}$ is homogeneous $\overset{\mathrm{def}}{\iff}$ Aut($\mathcal{K}$) acts transitively on $\mathrm{ri}\, \mathcal{K}$.

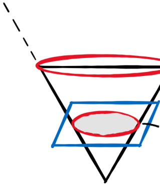
$\mathcal{K}$ is self-dual $\overset{\mathrm{def}}{\iff}$ $\mathcal{K} = \mathcal{K}^*$ for some choice of inner product.

$\mathcal{K}$ is symmetric $\overset{\mathrm{def}}{\iff}$ $\mathcal{K}$ is homogeneous and self-dual.

Recent results on amenability

A few results (L, Roshchina and Saunderson)

- Hyperbolicity cones and spectrahedral cones are amenable.
- Amenability is preserved by intersections and taking slices.
- A cone constructed from an amenable compact convex set is amenable.



L, V. Roshchina and J. Saunderson
Amenable cones are particularly nice.
arxiv:2011.07745

L, V. Roshchina and J. Saunderson
Hyperbolicity cones are amenable.
arxiv:2102.06359

Comparison of exposedness properties

Known results:

- Facially exposed $\Leftarrow$ Nice $\Leftarrow$ **Amenable** $\overset{\text{EPBR}}{\Leftarrow}$ Projectionally exposed.
- $\dim \mathcal{K} \leq 3$: Facially exposed $\Leftrightarrow$ Projectionally exposed (Barker and Poole, SIADM'87)
- There exists a 4D cone that is facially exposed but not nice (Vera, SIOPT'14).

New results (see LRS'20):

- There exists a 4D cone that is nice but not amenable
- In dimension 4 or less: Amenable $\Leftrightarrow$ Projectionally exposed.

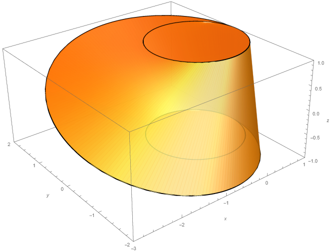


Figure: A 3D slice of a 4D convex cone that is nice but not amenable

Facial Residual Functions

Let

- $\mathcal{K}$: closed convex pointed cone.
- $\mathcal{F}$: face of $\mathcal{K}$
- $z \in \mathcal{F}^*$ (Reminder: $\mathcal{F}^* = \{x \mid \langle x, y\rangle \geq 0, \forall y \in \mathcal{F}\}$).

Fact:

$$\mathcal{F} \cap \{z\}^\perp = \mathcal{K} \cap \operatorname{span} \mathcal{F} \cap \{z\}^\perp.$$

Therefore,

$$\operatorname{dist}(x, \mathcal{K}) = 0 \quad \operatorname{dist}(x, \operatorname{span} \mathcal{F}) = 0 \quad \langle x, z\rangle = 0$$

implies

$$\operatorname{dist}(x, \mathcal{F} \cap \{z\}^\perp) = 0.$$

Facial Residual Functions

Let

- $\mathcal{K}$: closed convex pointed cone.
- $\mathcal{F}$: face of $\mathcal{K}$
- $z \in \mathcal{F}^*$, where $\mathcal{F}^* = \{x \mid \langle x, y \rangle \geq 0, \forall y \in \mathcal{F}\}$.

If

$$\operatorname{dist}(x, \mathcal{K}) \leq \epsilon \quad \operatorname{dist}(x, \operatorname{span} \mathcal{F}) \leq \epsilon \quad \langle x, z \rangle \leq \epsilon,$$

what can we say about

$$\operatorname{dist}(x, \mathcal{F} \cap \{z\}^{\perp})?$$

In general, it also depends on $\|x\|$.

Facial Residual Functions

Let

- $\mathcal{K}$: closed convex pointed cone.
- $\mathcal{F}$: face of $\mathcal{K}$
- $z \in \mathcal{F}^*$

Definition (Facial residual functions)

If $\psi_{\mathcal{F},z} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

1. $\psi_{\mathcal{F},z}$ is nonnegative, monotone nondecreasing in each argument and $\psi(0,\alpha) = 0$ for every $\alpha \in \mathbb{R}_+$.
2. whenever $x \in \operatorname{span} \mathcal{K}$ satisfies the inequalities

$$\operatorname{dist}(x, \mathcal{K}) \leq \epsilon, \quad \langle x, z \rangle \leq \epsilon, \quad \operatorname{dist}(x, \operatorname{span} \mathcal{F}) \leq \epsilon$$

   we have:

$$\operatorname{dist}(x, \mathcal{F} \cap \{z\}^\perp) \leq \psi_{\mathcal{F},z}(\epsilon, \|x\|).$$

Then, $\psi_{\mathcal{F},z}$ is said to be **facial residual function (FRF) for $\mathcal{F}$ and z**.

Fact: FRFs always exist!

Facial Residual Functions (FRFs) - Examples

- If $\mathcal{K}$ is a symmetric cone, then

$$\psi_{\mathcal{F},z}(\epsilon, \|x\|) = \kappa\epsilon + \kappa\sqrt{\epsilon\|x\|}$$

  is a FRF, for some $\kappa > 0$. (relatively technical to prove)
- If $\mathcal{K}$ is polyhedral, then $\psi_{\mathcal{F},z}(\epsilon, \|x\|) = \kappa\epsilon$ is a FRF, for some $\kappa > 0$.
- There are easy formulae for direct products of amenable cones and bijective linear images of cones.

An intermediary result

$$\begin{aligned} \text{find} \quad & x \\ \text{subject to} \quad & x \in (\mathcal{L} + a) \cap \mathcal{K} \end{aligned}$$

Proposition (Error bound for when a face satisfying Slater's condition is known)

Suppose $\mathcal{K}$ *is* amenable *and* $(\mathcal{L} + a) \cap \operatorname{ri} \mathcal{F} \neq \emptyset$ for some face $\mathcal{F}$ containing $(\mathcal{L} + a) \cap \mathcal{K}$.
Then, $\exists \kappa > 0$ *such that whenever* $x \in \operatorname{span} \mathcal{K}$ *and* ϵ *satisfy*

$$\operatorname{dist}(x, \mathcal{K}) \leq \epsilon, \quad \operatorname{dist}(x, \mathcal{L} + a) \leq \epsilon, \quad \operatorname{dist}(x, \operatorname{span} \mathcal{F}) \leq \epsilon,$$

we have

$$\operatorname{dist}(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq \kappa \|x\| \epsilon + \kappa \epsilon.$$

Fact: If $(\mathcal{L} + a) \cap \mathcal{K} \neq \emptyset$, a face $\mathcal{F}$ as above always exist.

- If we know $\mathcal{F}$ and we have a bound on $\operatorname{dist}(x, \operatorname{span} \mathcal{F})$, we can also bound $\operatorname{dist}(x, (\mathcal{L} + a) \cap \mathcal{K})$.

General idea

$$\begin{aligned} \text{find} \quad & x \\ \text{subject to} \quad & x \in (\mathcal{L} + a) \cap \mathcal{K} \end{aligned}$$

Suppose we have $\operatorname{dist}(x, \mathcal{L} + a)$ and $\operatorname{dist}(x, \mathcal{K})$.

1. Find $\mathcal{F}$ such that $(\mathcal{L} + a) \cap \operatorname{ri} \mathcal{F} \neq \emptyset$ and $(\mathcal{L} + a) \cap \mathcal{K} \subseteq \mathcal{F}$.
2. Use facial residual functions to bound $\operatorname{dist}(x, \mathcal{F})$ using $\operatorname{dist}(x, \mathcal{L} + a)$ and $\operatorname{dist}(x, \mathcal{K})$.
3. Use previous proposition!

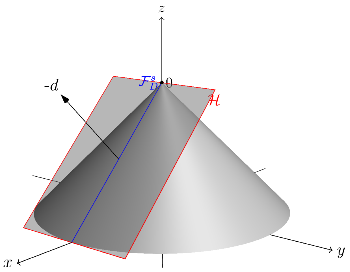
How to find $\mathcal{F}$

We want $\mathcal{F}$ such that

$$(\mathcal{L} + a) \cap \operatorname{ri} \mathcal{F} \neq \emptyset.$$

and $(\mathcal{L} + a) \cap \mathcal{K} \subseteq \mathcal{F}$.

1. Let $\mathcal{F}_1 = \mathcal{K}$ and $i \leftarrow 1$.
2. If $(\mathcal{L} + a) \cap \operatorname{ri} \mathcal{F}_i \neq \emptyset$, we are done.
3. If $(\mathcal{L} + a) \cap \operatorname{ri} \mathcal{F}_i = \emptyset$, we invoke a separation theorem.
   - There exists $z_i \in \mathcal{F}_i^* \setminus \mathcal{F}_i^\perp$ and $z_i \in \mathcal{L}^\perp \cap \{a\}^\perp$.
   - Let $\mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \cap \{z_i\}^\perp$ and $i \leftarrow i + 1$. Go to Step 2.



The Facial Reduction Theorem

Theorem (The facial reduction theorem)

If the problem is feasible, there exists a chain of faces

$$\mathcal{F}_\ell \subsetneq \cdots \subsetneq \mathcal{F}_1 = \mathcal{K}$$

together with $z_i \in \mathcal{F}_i^* \cap \mathcal{L}^\perp \cap \{a\}^\perp$ *such that*

1. *For all* $i \in \{1, \ldots, \ell - 1\}$, *we have*

$$\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{z_i\}^\perp$$

2. $\mathcal{F}_\ell \cap (\mathcal{L} + a) = \mathcal{K} \cap (\mathcal{L} + a)$ *and* $(\mathcal{L} + a) \cap \operatorname{ri} \mathcal{F}_\ell \neq \emptyset$.

The smallest ℓ is the **singularity degree** of the problem.

L, M. Muramatsu and T. Tsuchiya.
Facial reduction and partial polyhedrality.
SIAM Journal on Optimization, 28(3), 2018.

J. M. Borwein and H. Wolkowicz.
Facial reduction for a cone-convex programming problem.
Journal of the Australian Mathematical Society (Series A), 30(3):369–380, 1981.

Main result

Theorem (Error bound for amenable cones)

Let $\mathcal{K}$ be a closed convex amenable cone such that $\mathcal{K} \cap (\mathcal{L} + a) \neq \emptyset$. Let

$$\mathcal{F}_\ell \subsetneq \cdots \subsetneq \mathcal{F}_1 = \mathcal{K}$$

be a chain of faces of $\mathcal{K}$ together with $z_i \in \mathcal{F}_i^* \cap \mathcal{L}^\perp \cap \{a\}^\perp$ such that

$$(\mathcal{L} + a) \cap \operatorname{ri} \mathcal{F}_\ell \neq \emptyset.$$

and $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{z_i\}^\perp$ for every i. Let ψ_i be a facial residual function for $\mathcal{F}_i$, z_i. Then, after positive rescaling the ψ_i, there is a constant $\kappa > 0$ such that if $x \in \operatorname{span} \mathcal{K}$ satisfies the inequalities

$$\operatorname{dist}(x, \mathcal{K}) \leq \epsilon, \quad \operatorname{dist}(x, \mathcal{L} + a) \leq \epsilon,$$

we have

$$\operatorname{dist}\,(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq (\kappa\|x\| + \kappa)(\epsilon + \varphi(\epsilon, \|x\|)),$$

where $\varphi = \psi_{\ell-1} \diamondsuit \cdots \diamondsuit \psi_1$, if $\ell \geq 2$. If $\ell = 1$, we let φ be the function satisfying $\varphi(\epsilon, \|x\|) = \epsilon$.

$$(f \diamondsuit g)(a, b) := f(a + g(a, b), b).$$

Main result

Theorem (Error bound for amenable cones)

Let $\mathcal{K}$ be a closed convex amenable cone such that $\mathcal{K} \cap (\mathcal{L} + a) \neq \emptyset$. Let

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where $\varphi = \psi_{\ell-1} \diamondsuit \cdots \diamondsuit \psi_1$, if $\ell \geq 2$. If $\ell = 1$, we let φ be the function satisfying $\varphi(\epsilon, \|x\|) = \epsilon$.

$$(f \diamondsuit g)(a, b) := f(a + g(a, b), b).$$

Error bound for symmetric cones

Proposition (Error bounds for symmetric cones)

Let $\mathcal{K} = \mathcal{K}^1 \times \cdots \times \mathcal{K}^s$ *be a product of s symmetric cones, such that*

$$(\mathcal{L} + a) \cap \mathcal{K} \neq \emptyset$$

Let $\rho > 0$. *Then, there exists* $\kappa > 0$ *such that for every* $x \in \mathcal{E}$ *and* $\epsilon \leq 1$ *satisfying*

$$\operatorname{dist}(x, \mathcal{K}) \leq \epsilon, \quad \operatorname{dist}(x, \mathcal{L} + a) \leq \epsilon, \quad \|x\| \leq \rho,$$

we have

$$\operatorname{dist}\,(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq \kappa \epsilon^{(2^{-d_{PPS}(\mathcal{L},a)})}$$

Furthermore,

$$d_{PPS}(\mathcal{L}, a) \leq \min\left\{\dim(\mathcal{L}^\perp \cap \{a\}^\perp), \sum_{i=1}^{s}(\operatorname{rank} \mathcal{K}^i - 1), d(\mathcal{L}, a)\right\}.$$

- i.e., an uniform Hölderian error bound holds
- the exponent is $2^{-d_{\text{PPS}}(\mathcal{L},a)}$ and it is tight.

Conclusion so far

Three steps for obtaining error bounds.

1. Prove that $\mathcal{K}$ is *amenable*
2. Work out the *facial residual functions*. Preferably, the facial residual functions should have some simple formula as functions of ϵ and $\|x\|$.
3. Apply the main result.

Next questions

How to compute the facial residual functions? How about non-amenable cones?

Scott B. Lindstrom; L and Ting Kei Pong
Error bounds, facial residual functions and applications to the exponential cone
arXiv:2010.16391

Error bound without amenability

Theorem (Error bound without amenable cones, Lindstrom, L., Pong)

Let $\mathcal{K}$ be a closed convex cone such that $\mathcal{K} \cap (\mathcal{L} + a) \neq \emptyset$. Let

$$\mathcal{F}_\ell \subsetneq \cdots \subsetneq \mathcal{F}_1 = \mathcal{K}$$

be a chain of faces of $\mathcal{K}$ together with $z_i \in \mathcal{F}_i^* \cap \mathcal{L}^\perp \cap \{a\}^\perp$ such that

$$(\mathcal{L} + a) \cap \operatorname{ri} \mathcal{F}_\ell \neq \emptyset.$$

and $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{z_i\}^\perp$ for every i. Let ψ_i be a facial residual function for $\mathcal{F}_i$, z_i. Then, after positive rescaling the ψ_i, for every bounded set B there are constants $\kappa > 0$, $M > 0$ such that if $x \in \operatorname{span} \mathcal{K} \cap B$ satisfies the inequalities

$$\operatorname{dist}(x, \mathcal{K}) \leq \epsilon, \quad \operatorname{dist}(x, \mathcal{L} + a) \leq \epsilon,$$

we have

$$\operatorname{dist}\,(x, (\mathcal{L} + a) \cap \mathcal{K}) \leq \kappa(\epsilon + \varphi(\epsilon, M)),$$

where $\varphi = \psi_{\ell-1} \diamondsuit \cdots \diamondsuit \psi_1$, if $\ell \geq 2$. If $\ell = 1$, we let φ be the function satisfying $\varphi(\epsilon, \|x\|) = \epsilon$.

$$(f \diamondsuit g)(a, b) := f(a + g(a, b), b).$$

Error bound without amenable cones

Theorem (Error bound without amenable cones, Lindstrom, L., Pong)

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where $\varphi = \psi_{\ell-1} \diamondsuit \cdots \diamondsuit \psi_1$, if $\ell \geq 2$. If $\ell = 1$, we let φ be the function satisfying $\varphi(\epsilon, \|x\|) = \epsilon$.

$$(f \diamondsuit g)(a, b) := f(a + g(a, b), b).$$

Where amenability fits in this?

$\mathfrak{g} : \mathbb{R}_+ \to \mathbb{R}_+$: monotone nondecreasing function with $\mathfrak{g}(0) = 0$.

Definition ($\mathfrak{g}$-amenability)

$\mathcal{F} \trianglelefteq \mathcal{K}$ is $\mathfrak{g}$-*amenable* if for every bounded set B, there exists $\kappa > 0$ such that

$$\mathrm{dist}\,(x, \mathcal{F}) \leq \kappa \mathfrak{g}(\mathrm{dist}\,(x, \mathcal{K})), \quad \forall x \in (\mathrm{span}\,\mathcal{F}) \cap B.$$

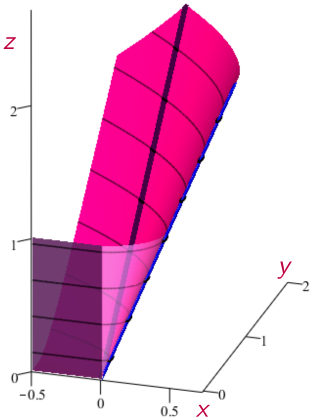
If all faces of $\mathcal{K}$ are $\mathfrak{g}$-amenable, then $\mathcal{K}$ is an $\mathfrak{g}$-*amenable cone*.

Suppose $\mathcal{K}^1$ and $\mathcal{K}^2$ are $\mathfrak{g}$-amenables

- There are calculus rules for the FRFs of $\mathcal{K}^1 \times \mathcal{K}^2$.
- A FRF of a **face** of $\mathcal{K}^1$ can be lifted to a FRF of the whole cone $\mathcal{K}^1$.
- Amenability is recovered when $\mathfrak{g} = |\cdot|$.
  - FRFs of $\mathcal{K}^1 \times \mathcal{K}^2$. are sums of FRFs of $\mathcal{K}^1$ and $\mathcal{K}^2$.

The exponential cone

$$K_{\exp} := \left\{(x, y, z) \mid y > 0, z \geq ye^{x/y}\right\} \cup \{(x, y, z) \mid x \leq 0, z \geq 0, y = 0\}.$$



The exponential cone

$$K_{\exp} := \left\{(x,y,z) \mid y>0, z \geq y e^{x/y}\right\} \cup \{(x,y,z) \mid x \leq 0, z \geq 0, y = 0\}.$$

$$K_{\exp}^* := \left\{(x,y,z) \mid x<0, ez \geq -x e^{y/x}\right\} \cup \{(x,y,z) \mid x = 0, ez \geq 0, y \geq 0\}.$$

1. **Not exposed!** (So not amenable...)
2. Applications to entropy optimization, logistic regression, geometric programming and etc.
3. Available in Mosek. https://docs.mosek.com/modeling-cookbook/expo.html.

V. Chandrasekaran, P. Shah
Relative entropy optimization and its applications.
Math. Program. 161, 1–32 (2017)

The faces of the exponential cone

(a) exposed extreme rays (1D faces) parametrized by $\beta \in \mathbb{R}$:

$$\mathcal{F}_\beta := \left\{ \left(-\beta y + y, y, e^{1-\beta} y\right) \;\middle|\; y \in [0, \infty) \right\}. \qquad \text{(amenable)}$$

(b) an "exceptional" exposed extreme ray:

$$\mathcal{F}_\infty := \{(x, 0, 0) \mid x \leq 0\}. \qquad \text{(amenable)}$$

(c) a **non-exposed** extreme ray: $\mathcal{F}_{ne}$:

$$\mathcal{F}_{ne} := \{(0, 0, z) \mid z \geq 0\}. \qquad \text{($\mathfrak{g}$-amenable, not amenable)}$$

(d) a single 2D exposed face:

$$\mathcal{F}_{-\infty} := \{(x, y, z) \mid x \leq 0, z \geq 0, y = 0\}, \qquad \text{(amenable)}$$

where $\mathcal{F}_\infty$ and $\mathcal{F}_{ne}$ are the extreme rays of $\mathcal{F}_{-\infty}$.

Error bound for problems over the exponential cone

$$\begin{aligned} &\text{find} \quad x \\ &\text{subject to} \quad x \in (\mathcal{L} + a) \cap K_{\exp} \end{aligned} \qquad \text{(CFP)}$$

Let $z \in (K_{\exp})^* \cap \mathcal{L}^\perp \cap \{a\}^\perp$, $z \neq 0$. Let $\mathcal{F} = K_{\exp} \cap \{z\}^\perp$.

- $\mathcal{F} = \{0\}$: Lipschitzian error bound.
- $\mathcal{F} = \mathcal{F}_\beta$: a Hölderian error bound with exponent 1/2.
- $\mathcal{F} = \mathcal{F}_\infty$, either a Lipschitzian or a log-type error bound holds depending on the exposing vector. (**No Hölderian error bound holds in the latter!**)
- $\mathcal{F} = \mathcal{F}_{-\infty}$, **entropic error bound**: for every bounded set B, there exists $\kappa_B > 0$

  $$\operatorname{dist}(x, (\mathcal{L} + a) \cap K_{\exp}) \le \kappa_B \mathfrak{g}_{-\infty}(\max(\operatorname{dist}(x, \mathcal{L} + a), \operatorname{dist}(x, K_{\exp}))), \quad \forall x \in B,$$

  where

  $$\mathfrak{g}_{-\infty}(t) := \begin{cases} 0 & \text{if } t = 0, \\ -t\ln(t) & \text{if } t \in \left(0, 1/e^2\right], \\ t + \frac{1}{e^2} & \text{if } t > 1/e^2. \end{cases}$$

The results above are **optimal**.

Strange error bounds

From the exponential cone we can:

- Obtain sets that **do not have** a Hölderian error bound, but have a logarithmic error bound:
    - Or, a function that does not have a KL exponent.

$$\mathcal{F}_\infty = K_{\exp} \cap \{z\}^\perp,$$

  where $z = (0, 0, 1)$.

- Obtain sets that satisfy a Hölderian bound for all $\gamma \in (0, 1)$ but not $\gamma = 1$. Furthermore, the best error bound is an entropic one.
    - Or, a KL function whose exponent can be arbitrary close to $1/2$ but not $1/2$.

$$\mathcal{F}_{-\infty} = K_{\exp} \cap \{z\}^\perp,$$

  where $z = (0, 1, 0)$.

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