Variational Analysis and Optimisation Webinair

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Tweaking Ramanujan's Approximation of $n!$

by

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&

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This presentation is based on my book

“Probabilty Theory Without Tears”

which is free to download from

www.ProbabilityTheoryWithoutTears.net

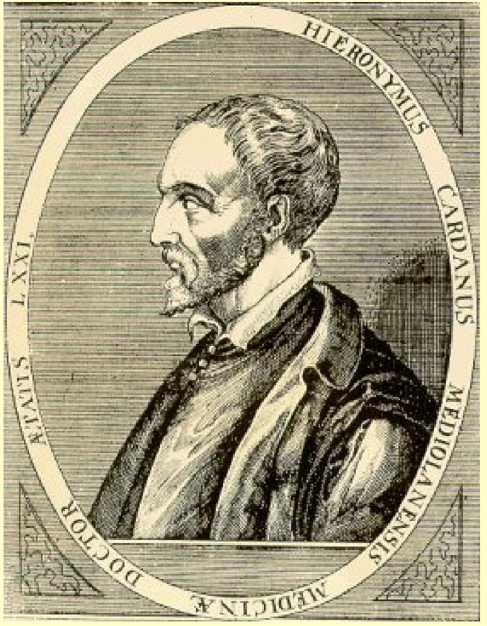
and my paper

“Tweaking Ramanujan’s Approximation of n!”

arXiv:2010.15512 [math.NT]

Introduction

The first book about games of chance "Liber de ludo aleae" (On Casting the Die) (with a section on effective cheating methods), was written by the Italian physician and one of the most influential Renaissance mathematicians Girolamo Cardano (1501–1576) in the 1560s, but not published until 1663. In fact he supported himself through medical school on winnings from gambling using his understanding of probability.



Cardano

In 1545 Cardano published “Ars Magna” (The Great Art) - the most important book on algebra in Latin. A second edition appeared in 1570. It is regarded as one of the most important scientific works of the early Renaissance period. It contains the first published method of solving cubic and quartic polynomial equations (discovered respectively by Scipione del Ferro (1465–1526) and Lodovico Ferrari (1522–1565), a student of Cardano). Ars Magna contains the first occurence of complex numbers. Of more interest to us today is the inclusion in Ars Magna of the topic: binomial coefficients and the binomial theorem.

In 1654 Antoine Gombauda, a French writer and nobleman, called to the attention of French mathematician Blaise Pascal (1623–1662) a problem concerning a popular dice game. The problem was about whether or not to bet even money on the occurrence of at least one "double six" during the 24 throws of a pair of dice.

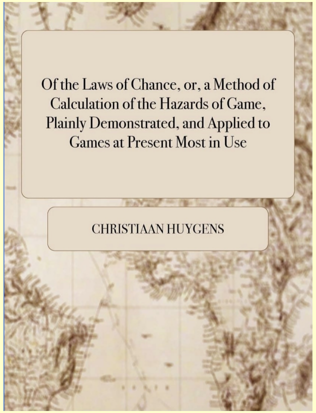
This led to an exchange of letters over several weeks between Pascal and the French mathematician Pierre de Fermat (1607–1665) and early steps in the development of the mathematical theory of probability.

Pascal

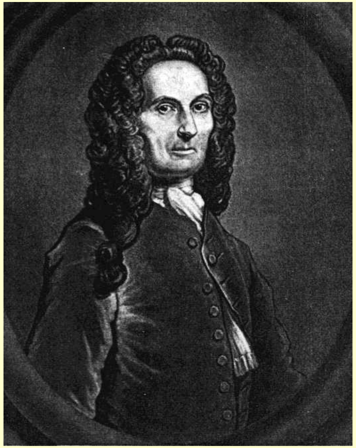
Fermat

The scientist Christian Huygens (1629–1695), a teacher of the German polymath Gottfried Wilhelm Leibniz (1646–1716) learned about the development of the mathematical theory of probability by Pascal and Fermat and in 1657 published “De Ratiociniis in Ludo Aleae” (The Value of Chance in Games of Fortune) the first printed book on probability. It included the notion of expected value. Huygens invented the pendulum clock and discovered Saturn’s moon Titus.

Huygens



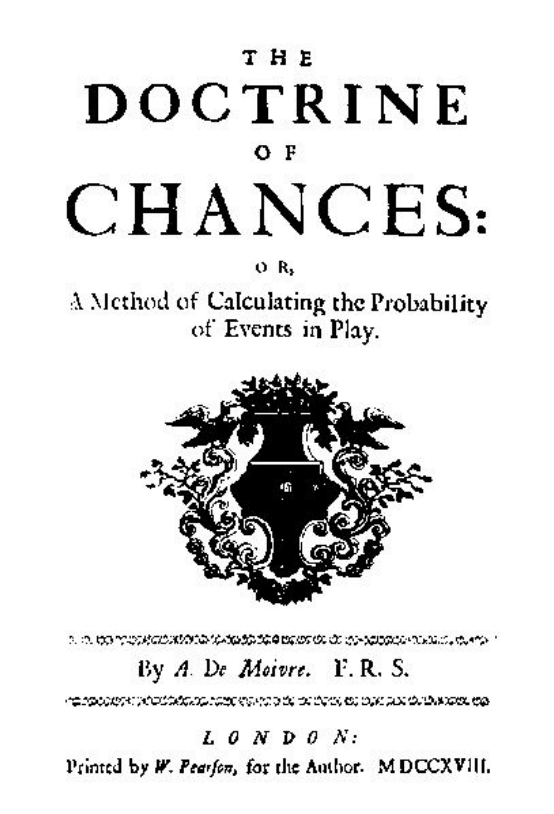
Abraham de Moivre (1667–1754) was a French Protestant who studied mathematics in Paris. He read and was influenced by Huygens book on probability and later Newton's Principia Mathematica. Because of the persecution of French Protestants, he sought asylum in England, where he lived for the rest of his life, earning his living as a private mathematics tutor and later as a consultant to gamblers and insurance brokers. He became a friend of Edmond Halley (1656–1742) and later Isaac Newton (1643–1727) and James Stirling (1692–1770).



de Moivre

The second printed book on probability, "The Doctrine of Chances", was published in 1718 by Abraham de Moivre. In his research on games he needed a good approximation for $\binom{2n}{n}$, where n may be quite large.

He proved $n!$ can be approximated by $cn^{n+\frac{1}{2}}e^{-n}$, for some constant c, where he gave an approximation for c (about 2.5).



THE
DOCTRINE
OF
CHANCES:
OR,
A Method of Calculating the Probability
of Events in Play.

By *A. De Moivre.* F. R. S.

LONDON:
Printed by *W. Pearson,* for the Author. MDCCXVIII.

About 1730 the Scottish mathematician James Stirling (1692–1770) proved that the constant c is $\sqrt{2\pi}$.

Stirling's Approximation is $\boxed{n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n}$. It is very good for large n but, as we see in the next table and graph, surprisingly it is quite good even for small n.

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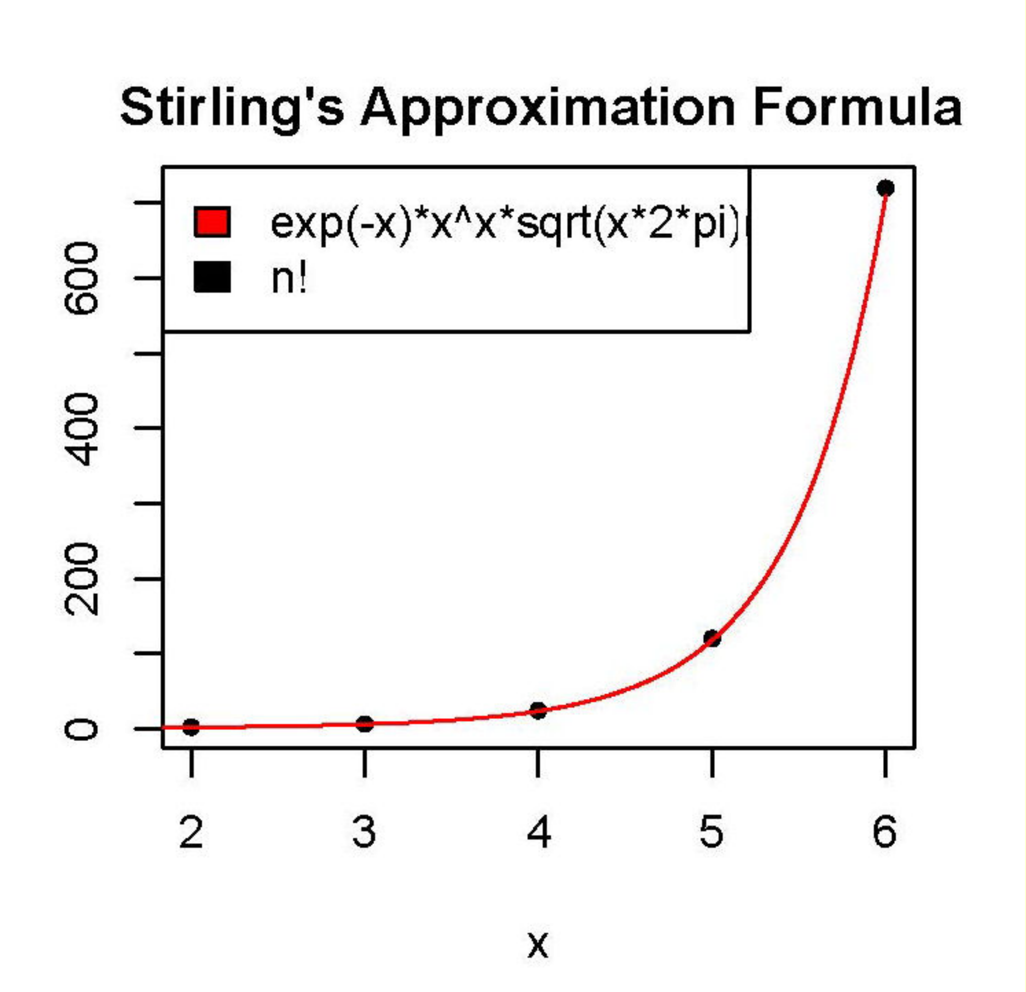
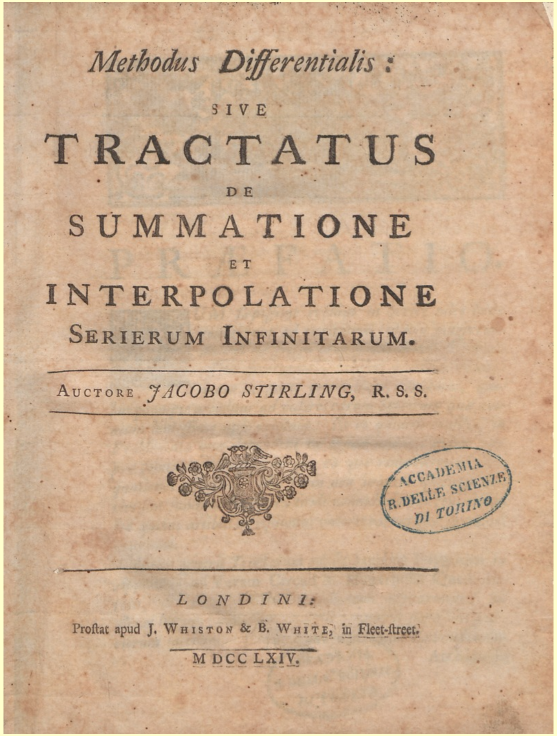
n	$n!$	Stirling's approx.	% error
2	2	1.919004	4.05
5	120	118.0192	1.65
10	3628800	3598696	0.83
20	2.432902×10^{18}	2.422787×10^{18}	0.42
50	3.041409×10^{64}	3.036345×10^{64}	0.17
100	9.332622×10^{157}	9.324848×10^{157}	0.08


Stirling's Approximation Formula
exp(-x)*x^x*sqrt(x*2*pi)
n!
0
200
400
600
2
3
4
5
6
x

Plaque to the Stirlings
in the garden of Dunblane Cathedral

Stirling's Approximation was published in 1764 in his book "Methodus differentialis" pictured below.



Example. Assume that in a particular country there were exactly one million births in 2020. Also assume that the probability that a baby is born biologically male is 0.5. What is the probability in 2020 there will be exactly 500,000 babies born which are biologically male? Use Stirling's formula to approximate this number.

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Clearly the probability is $\binom{1000000}{500000}(\frac{1}{2})^{1000000}$.

These numbers are rather big for a calculator.

So let us use Stirling's formula which says that

$$n! \sim (n^n/e^n)\sqrt{2\pi n}.$$

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So let us use Stirling formula which says that

$$n! \sim (n^n/e^n)\sqrt{2\pi n}.$$

$$\binom{2n}{n} = \frac{(2n)!}{n!.n!} \sim \frac{\frac{(2n)^{2n}}{e^{2n}}\sqrt{2\pi.2n}}{\frac{n^n}{e^n}\sqrt{2\pi n}.\frac{n^n}{e^n}\sqrt{2\pi n}} = \frac{2^{2n}}{\sqrt{\pi}\sqrt{n}}$$

$$\text{Thus} \quad \binom{2n}{n}.2^{-2n} \sim \frac{1}{\sqrt{\pi.n}}.$$

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$$\text{So} \quad \binom{1000000}{500000}.2^{-1000000} \sim \frac{1}{\sqrt{\pi.500000}}$$
$$= 0.00079788\ldots.$$

We observe firstly, that Stirling's formula avoided having to calculate 1000000!, and $(500000!)^2$. Secondly we note that the probability that exactly half of those born were biologically male is very small. Finally I mention that the actual value is indeed 0.00079788.... ■

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The extraordinary Indian mathematician Srinivasa Ramanujan (1887–1920) in the last year of his life gave a remarkably better asymptotic formula. This result became known when it appeared in 1988 in "The lost notebook and other unpublished papers", edited by S. Raghavan and S. S. Rangachari.

Ramanujan did not include a proof of his approximation claim.

Ramanujan's approximation is substantially better than all those which were published in the subsequent 80 years. For example, when n equals one million, the percentage error of Ramanujan's approximation is one million million times better than Gosper's.

In 2013 the Australian mathematician Michael Hirschhorn (from UNSW) and the Costa Rican mathematician Mark B. Villarino proved the correctness of Ramanujan's claim above (at least for positive integers).

Hirschhorn

In recent years there have been several improvements of Stirling's formula including by Nemes (in 2010), Windschitl (in 2002), and Chen (in 2016).

In this presentation today it is shown:

(i) how all these asymptotic results can be easily verified;

(ii) how Hirschhorn and Villarino's argument allows a tweaking of Ramanujan's result to give a better approximation;

(iii) that a new asymptotic formula can be obtained by further tweaking of Ramanujan's result;

(iv) that Chen's asymptotic formula is better than the others mentioned here, and the new asymptotic formula is comparable with Chen's.

The problem of extending the factorial from the positive integers to a wider class of numbers was first investigated by the Swiss mathematician Daniel Bernoulli (1700–1782) and the German mathematician Christian Goldbach (1690–1764) in the 1720s.

In 1729 Leonhard Euler (1707–1783) succeeded and in 1730 he proved that **for x any positive real number,**

$$\Gamma(x) = \int_0^\infty t^{x-1} e^t \, dt,$$

where $\Gamma(n) = (n-1)!$, for any positive integer n. (Indeed this is true for complex numbers with positive real part.)

Theorem 1. [Laplace's extension of Stirling's Formula to the Gamma Function]
(Pierre-Simon, marquis de Laplace (1749–1827))
For x a positive real number,

$$\Gamma(x+1) \sim (x^x/e^x)\sqrt{2\pi x},$$

$$\text{i.e.} \quad \lim_{x\to\infty} \frac{\Gamma(x+1)}{(x^x/e^x)\sqrt{2\pi x}} = 1.$$

In particular, for n a positive integer,

$$n! \sim (n^n/e^n)\sqrt{2\pi n}.$$

$$\text{i.e.} \quad \lim_{x\to\infty} \frac{\Gamma(n+1)}{(n^n/e^n)\sqrt{2\pi n}} = 1.$$

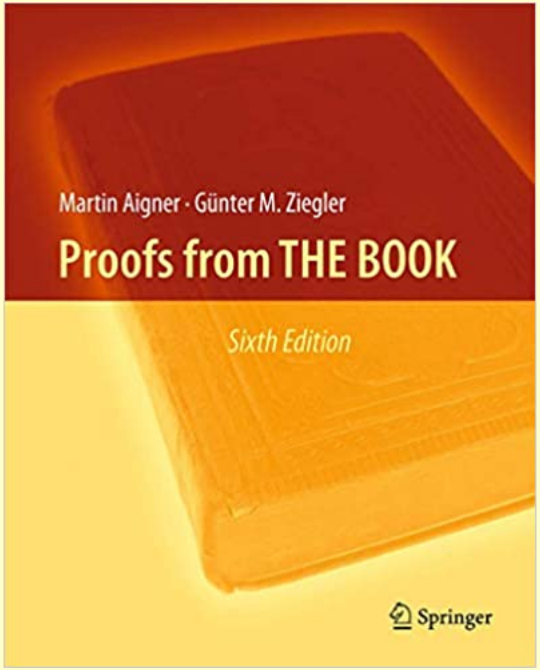
There are often many proofs of a theorem. Some proofs are described as short, others as informative, and yet others as elementary. Short is self-explanatory. An informative proof is one which helps you understand the theorem.

Erdös

An elementary proof is one which uses minimum background knowledge. **R. Michel's 2008 proof presented here of Stirling's Formula for the Gamma Function is elementary**, but not informative.

The Hungarian mathematician Paul Erdös (1913–1996) coined the term "Proofs from The Book".

"Proofs from THE BOOK is a book of mathematical proofs by Martin Aigner and Günter M. Ziegler. The book is dedicated to the mathematician Paul Erdös who often referred to 'The Book' in which God keeps the most elegant proof of each mathematical theorem."



Erdös said "You don't have to believe in God, but you should believe in The Book." The book is illustrated by Karl Heinrich Hofmann and is translated into several languages.

Before proceeding to the proof of Theorem 1 we need to recall that $\int\limits_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$

This is called the Gaussian integral after the German mathematician Johann Carl Friedrich Gauss (1777–1855) who published this integral in 1809. It is also known as the Euler-Poisson integral after Euler and the French mathematician Siméon Denis Poisson (1781–1840). It was first evaluated by Laplace using a change of variable.

On the Eiffel Tower 72 names of eminent French STEM researchers and scholars are engraved, including Poisson, Laplace, Napier, Cauchy, Lagrange, and Coriolis.

Proof of Theorem 1. Let $x, t \in \mathbb{R}$, $t, x > 0$. Further, let $f(x) = x^t e^{-x}$ and $A = \{x : |x - t| \geqslant \frac{t}{2}\}$. Let g_A be the characteristic function of A; that is, $g_A(x) = 1$, for $x \in A$ and $g_A(x) = 0$, otherwise. As $\Gamma(t+1) = \int_0^\infty f(x)\,dx$ we see

$$\Gamma(t+1) = \int_{\frac{t}{2}}^{\frac{3t}{2}} f(x)\,dx + \int_0^\infty g_A(x) f(x)\,dx \tag{1}$$

Proof. Let $x, t \in \mathbb{R}$, $t, x > 0$. Further, let $f(x) = x^t e^{-x}$ and $A = \{x : |x-t| \geqslant \frac{t}{2}\}$. Let g_A be the characteristic function of A; that is, $g_A(x) = 1$, for $x \in A$ and $g_A(x) = 0$, otherwise. As $\Gamma(t+1) = \int_0^\infty f(x)\,dx$ we see

$$\Gamma(t+1) = \int_{\frac{t}{2}}^{\frac{3t}{2}} f(x)\,dx + \int_0^\infty g_A(x) f(x)\,dx \qquad (1)$$

Observe that: $$x \in A \implies 1 \leqslant \frac{4(x-t)^2}{t^2}; \qquad (2)$$

$$\Gamma(z+1) = z\Gamma(z), \text{ for all } z \in \mathbb{R}, z > 0 : \qquad (3)$$

$$\Gamma(t+2) = (t+1)\Gamma(t+1); \qquad (4)$$

$$\Gamma(t+3) = (t+2)(t+1)\Gamma(t+1). \qquad (5)$$

$$\left|1 - \frac{1}{\Gamma(t+1)}\int_{\frac{t}{2}}^{\frac{3t}{2}} x^t e^{-x}\, dx\right| = \left|1 - \frac{1}{\Gamma(t+1)}\left(\Gamma(t+1) - \int_0^\infty g_A(x) x^t e^{-x}\, dx\right)\right|$$

$$\leqslant \left|1 - \frac{1}{\Gamma(t+1)}\left(\Gamma(t+1) - \int_0^\infty \frac{4(x-t)^2}{t^2} x^t e^{-x}\, dx\right)\right|$$

$$= \left|\frac{4}{t^2\Gamma(t+1)}\int_0^\infty (x^2 + t^2 - 2xt) x^t e^{-x}\, dx\right|$$

$$= \left|\frac{4}{t^2\Gamma(t+1)}\left(\int_0^\infty x^2 x^t e^{-x}\, dx + \int_0^\infty t^2 x^t e^{-x}\, dx - \int_0^\infty 2xt x^t e^{-x}\, dx\right)\right|$$

$$= \left|\frac{4}{t^2\Gamma(t+1)}(\Gamma(t+3) + t^2\Gamma(t+1) - 2t\Gamma(t+2))\right|$$

$$= \left|\frac{4}{t^2\Gamma(t+1)}((t+2)(t+1)\Gamma(t+1) + t^2\Gamma(t+1) - 2t(t+1)\Gamma(t+1))\right|$$

$$= \frac{4}{t^2}(t^2 + 3t + 2 + t^2 - 2t^2 - 2t) = \frac{4}{t^2}(t+2).$$

Therefore $$\lim_{t\to\infty} \frac{1}{\Gamma(t+1)}\int_{\frac{t}{2}}^{\frac{3t}{2}} x^t e^{-x}\, dx = 1. \tag{6}$$

So we have $$\lim_{t\to\infty} \frac{1}{\Gamma(t+1)} \int_{\frac{t}{2}}^{\frac{3t}{2}} x^t e^{-x}\, dx = 1. \qquad (6)$$

Make the change of variables $x = y\sqrt{t} + t$ and define $h_t(y) = \left(1 + \frac{y}{\sqrt{t}}\right)^t e^{-y\sqrt{t}}$. So

$$x = \frac{t}{2} \iff y = -\frac{\sqrt{t}}{2} \quad \text{and } x = \frac{3t}{2} \iff y = \frac{\sqrt{t}}{2}. \qquad (7)$$

Now $$\int_{-\frac{\sqrt{t}}{2}}^{\frac{\sqrt{t}}{2}} h_t(y)\,dy = \int_{-\frac{\sqrt{t}}{2}}^{\frac{\sqrt{t}}{2}} \left(1+\frac{y}{\sqrt{t}}\right)^t e^{-y\sqrt{t}}\,dy$$

$$= \int_{\frac{t}{2}}^{\frac{3t}{2}} \left(1+\frac{x-t}{\sqrt{t}\sqrt{t}}\right)^t e^{-\left(\frac{x-t}{\sqrt{t}}\right)\sqrt{t}}\,\frac{1}{\sqrt{t}}\,dx$$

$$= \frac{e^t}{t^t\sqrt{t}}\int_{\frac{t}{2}}^{\frac{3t}{2}} x^t e^{-x}\,dx.$$

By (6) and (7) this implies

$$\lim_{t\to\infty} \frac{t^t\sqrt{t}}{\Gamma(t+1)\,e^t} \int_{-\frac{\sqrt{t}}{2}}^{\frac{\sqrt{t}}{2}} h_t(y)\,dy = 1. \qquad (8)$$

Now $$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \text{ for } |x| < \frac{1}{2}. \qquad (9)$$

So by (9) for $z \in \mathbb{R}$ with $|z| < \frac{1}{2}$,

$$\left|\ln(1+z) - z + \frac{1}{2}z^2\right| \leqslant \sum_{n=3}^{\infty} \frac{|z|^n}{n} \leqslant \sum_{n=3}^{\infty} \frac{|z|^n}{3}$$

$$= \frac{|z|^3}{3} \sum_{n=3}^{\infty} |z|^{n-3} = \frac{|z|^3}{3} \frac{1}{1-|z|} \leqslant \frac{2}{3}|z|^3. \qquad (10)$$

So for all $u, v \in \mathbb{R}$, we use Taylor series

We have: $\left|\ln(1+z) - z + \frac{1}{2}z^2\right| \leqslant \frac{2}{3}|z|^3. \qquad (10)$

$$|e^u - e^v| = e^v|e^{u-v} - 1| = e^v \left|\sum_{n=1}^{\infty} \frac{u-v)^n}{n!}\right|$$

$$\leqslant e^v|u-v| \sum_{n=1}^{\infty} \frac{|u-v|^{n-1}}{n!} \leqslant e^v|u-v|e^{|u-v|} \qquad (11)$$

In (10) put $z = \frac{y}{\sqrt{t}}$ to obtain $\left|\ln\left(1 + \frac{y}{\sqrt{t}}\right) - \frac{y}{\sqrt{t}} + \frac{y^2}{2t}\right| \leqslant \frac{2|y|^3}{3\,t^{\frac{3}{2}}}. \qquad (12)$

Put $u = \ln h_t(y) = \ln\left(\left(1 + \frac{y}{\sqrt{t}}\right)^t e^{-y\sqrt{t}}\right) = t(\ln(1+z) - z). \qquad (13)$

Put $v = -\frac{y^2}{2} = -\frac{tz^2}{2}$. Then for $|z| < \frac{1}{2} \iff |y| < \frac{\sqrt{t}}{2}$, we have

$$\left|h_t(y) - e^{-\frac{y^2}{2}}\right| = \left|e^{t(\ln(1+z)-z)} - e^{-\frac{tz^2}{2}}\right|, \quad \text{by (13)}$$

$$\leqslant e^{-\frac{tz^2}{2}} \left|t(\ln(1+z)-z) + \frac{tz^2}{2}\right| e^{|t(\ln(1+z)-z)+\frac{tz^2}{2}|}, \quad \text{by (11)}$$

$$\leqslant e^{-\frac{y^2}{2}} \frac{t2|y|^3}{3t^{\frac{3}{2}}} e^{\left(\frac{2t|y|^3}{3t^{3/2}}\right)}, \quad \text{by (12)}$$

$$\leqslant \frac{|y|^3}{\sqrt{t}} \frac{2}{3} e^{y^2(-1/2+\frac{2ty}{3t^{3/2}})}$$

$$< \frac{|y|^3}{\sqrt{t}} \frac{2}{3} e^{y^2(-1/2+\frac{2t\sqrt{t}}{2.3.t^{3/2}})}, \quad \text{as } y < \frac{\sqrt{t}}{2}$$

$$< \frac{|y|^3}{\sqrt{t}} e^{\frac{-y^2}{6}} \qquad (14)$$

Therefore, by (14),

$$\left|\int_{\frac{-\sqrt{t}}{2}}^{\frac{\sqrt{t}}{2}} h_t(y)\,dy - \int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right|$$

$$\leqslant \frac{1}{\sqrt{t}}\int_{\frac{-\sqrt{t}}{2}}^{\frac{\sqrt{t}}{2}} |y|^3 e^{-y^2/6}\,dy + \int_{|y|>\sqrt{t}/2} e^{-y^2/2}\,dy$$

But the first integral on the right hand side is finite as it is less than $\int_{-\infty}^{\infty} |y|^3 e^{-y^2/6}\,dy$ which, substituting $u = y^2$ and integrating by parts, is easily proved to equal 36. So the limit as $t \to \infty$ of the first term on the right hand side is zero.

Therefore

$$\lim_{t\to\infty}\int_{\frac{-\sqrt{t}}{2}}^{\frac{\sqrt{t}}{2}} h_t(y)\,dy = \int_{-\infty}^{\infty} e^{-y^2/2}\,dy = \sqrt{2\pi}$$

Recalling (8)

$$\lim_{t\to\infty}\frac{t^t\sqrt{t}}{\Gamma(t+1)\,e^t}\int_{-\frac{\sqrt{t}}{2}}^{\frac{\sqrt{t}}{2}} h_t(y)\,dy = 1. \qquad (8)$$

So combining the above with (8) we have

$$\lim_{t\to\infty}\frac{\Gamma(t+1)e^t}{\sqrt{2\pi}t^{(t+1/2)}} = 1.$$

This completes the proof of the theorem giving Stirling's Formula for the Gamma Function. ■

Most of the proofs in the literature of Stirling's formula and its extensions prove that they are asymptotic by establishing an error estimate such as

$$\Gamma(x+1) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + O\left(x^{-1}\right)\right).$$

In fact most of the effort goes into proving such error estimates.

In this paper we observe that once one knows that Stirling's formula is asymptotic to $\Gamma(x+1)$, all of the other known asymptotic formulae can be verified trivially without the need to establish any error estimates.

In 1917 the well-known group theorist William Burnside (1852–1927) published a modest improvement on Stirling's formula, namely

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+1/2}.$$

How modest an improvement it is can be ascertained from Table 1 below.

In 1978 Ralph William (Bill) Gosper Jr published a significant improvement on Stirling and Burnside's formulae. It was that

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^{x}\sqrt{2x+\frac{1}{3}}.$$

In a web post in 2002, Robert H. Windschitl gave an elegant and good asymptotic approximation of $n!$, namely that

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\left(\frac{1}{x}\right)\right)^{\frac{x}{2}}.$$

In 2010 Gergő Nemes gave an asymptotic approximation which is almost as good as Windschitl's but better than all the others at that time. It was that

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^2 - \frac{1}{10}}\right)^x.$$

An asymptotic formula of a different style, which is much better than Gosper's, was published in 2011 by Cristinel Mortici . It was

$$\Gamma(x+1) \sim \sqrt{2\pi\, x}\left(\frac{x}{e} + \frac{1}{12\, e\, x}\right)^{x}.$$

Pierre-Simon Laplace (1749–1827) discovered what is now known as the Stirling series for the gamma function.

$$\Gamma(x+1) \sim e^{-x}x^{x+\frac{1}{2}}\sqrt{2\pi}\left(1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51840x^3}\right.$$
$$\left.-\frac{571}{2488320x^4}+\sum_{n=5}^{\infty}\frac{a_n}{b_n x^n}\right),$$

where the real numbers a_n and b_n are explicitly calculated in a 2010 paper by G. Nemes. As stated by V. Namias, "the performance deteriorates as the number of terms is increased beyond a certain value".
In Table 2 we show how using up to the term x^{-4} in this divergent series compares with the other approximations.

A major advance in producing an asymptotic formula for $n!$ was made by the Indian mathematician Srinivasa Ramanujan (1887–1920) in the last year of his life. Ramanujan's claim, was that

$$\Gamma(x+1) = \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{\theta_x}{30}\right)^{\frac{1}{6}},$$

where $\theta_x \to 1$ as $x \to \infty$ and $\frac{3}{10} < \theta_x < 1$. He gave numerical evidence for his claim.

Ramanujan's approximation is substantially better than those published in the subsequent 80 years. For example, when $n = 1$ million, the percentage error of Ramanujan's approximation is one million million times better than Gosper's.

In 2013 Michael Hirschhorn and Mark B. Villarino proved Ramanujan's claim above for positive integers. They showed that Ramanujan's θ_n satisfies for each positive integer n:

$$1-\frac{11}{8n}+\frac{79}{112n^2}<\theta_n<1-\frac{11}{8n}+\frac{79}{112n^2}+\frac{20}{33n^3}.$$

Without explicitly saying it, it is clear from their work that $n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n\left(8n^3+4n^2+n+\frac{1-\frac{11}{8n}+\frac{79}{112n^2}}{30}\right)^{\frac{1}{6}}$,

for positive integers n. This approx. is better than all that preceded it. For $n = 1$ million, it has a percentage error at least one million times better than each one.

In 2016 Chao-Ping Chen produced an asymptotic approximation which for $n = 1$ million has a percentage error one million times better than that of Hirschhorn and Villarino. His asymptotic approximation is

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right)^{x^2+\frac{53}{210}}.$$

A more detailed analysis of Hirschhorn and Villarino's improvement on that of Ramanujan, suggests a tweaking of their approximation. That tweaking produces an approximation which is stated in our final Corollary and is comparable to Chen's for $n = 1$ to $n = 10{,}000$ and better than Chen's for $n = 1$ million.

Theorem 1. Let $\boldsymbol{f}$ be a function from a subset $(\boldsymbol{a}, \infty)$ to $\mathbb{R}$, where $\boldsymbol{a} \in \mathbb{R}, \boldsymbol{a} > \mathbf{0}$.
If $\lim_{x \to \infty} f(x) = 1$, then $\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x . f(x)$.

Proof. This follows immediately from the Stirling approximation, namely that

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x .$$

As an immediate corollary of Theorem 1 we obtain that all of the other mentioned approximations are asymptotic to $\Gamma(x+1)$. Some of these were proved by the authors only for x a positive integer.

Corollary. For x a positive real number:

(i) Burnside: $\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+1/2}$;

(ii) Gospe: $\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^{x}\sqrt{2x+\frac{1}{3}}$;

(iii) Mortici: $\Gamma(x+1) \sim \sqrt{2\pi\, x}\left(\frac{x}{e}+\frac{1}{12\, e\, x}\right)^{x}$;

(iv) Ramanujan: $\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^{x}\left(8x^3+4x^2+x+\frac{1}{30}\right)^{\frac{1}{6}}$;

(v) Laplace (n): Fix $n \in \mathbb{N}$. For $a_i, b_i \in \mathbb{N}$,

$$\Gamma(x+1) \sim e^{-x}x^{x+\frac{1}{2}}\sqrt{2\pi}\left(1+\frac{1}{12x}+\frac{1}{288x^2}+\sum_{i=3}^{n}\frac{a_i}{b_i x^i}\right);$$

(vi) Nemes: $\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^{x}\left(1+\frac{1}{12x^2-\frac{1}{10}}\right)^{x}$.

(vii) Windschitl: $\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^{x}\left(x\sinh\left(\frac{1}{x}\right)\right)^{\frac{x}{2}}$.

(viii) Hirschhorn & Villarino :

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^{x}\left(8x^3+4x^2+x+\frac{1-\frac{11}{8x}+\frac{79}{112x^2}}{30}\right)^{\frac{1}{6}}.$$

(ix) Chen: $\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^{x}\left(1+\frac{1}{12x^3+\frac{24}{7}x-\frac{1}{2}}\right)^{x^2+\frac{53}{210}}$.

Proof. In each case it is sufficient to determine the function f in Theorem 1 and observe that $\lim_{x \to \infty} f(x) = 1$.

(i) Use $f(x) = \left(1 + \frac{1}{2x}\right)^x \left(\frac{1 + \frac{1}{2x}}{e}\right)^{\frac{1}{2}}$.

(ii) Use $f(x) = \sqrt{1 + \frac{1}{6x}}$.

(iii) Use $f(x) = \left(1 + \frac{1}{12x^2}\right)^x$.

(iv) Use $f(x) = \left(1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1}{240x^3}\right)^{\frac{1}{6}}$.

(v) Use $f(x) = \left(1 + \frac{1}{12x} + \frac{1}{288x^2} + \sum_{i=3}^{n} \frac{a_i}{b_i x^i}\right)$.

(vi) Use $f(x) = \left(1 + \frac{1}{12x^2 - \frac{1}{10}}\right)^x$.

(vii) Use $f(x) = \left(x \sinh\left(\frac{1}{x}\right)\right)^{\frac{x}{2}}$.

(viii) Use $f(x) = \left(1 + \frac{1}{2x} + \frac{1}{8x^2} + \frac{1 - \frac{11}{8x} + \frac{79}{112x^2}}{240x^3}\right)^{\frac{1}{6}}$.

(ix) Use $f(x) = \left(1 + \frac{1}{12x^3 + \frac{24}{7}x - \frac{1}{2}}\right)^{x^2 + \frac{53}{210}}$.

An even better approximation for x equals 1 million which we refer to in the table below as the SAM approximation is:

Corollary. For x a positive real number,

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3+4x^2+x+\frac{1-\frac{11}{8x}+\frac{79}{112x^2}+\frac{A}{x^3}}{30}\right)^{\frac{1}{6}},$$

where $A = \dfrac{380279456577}{722091376690}$. ∎

The tables in this section were calculated using the WolframAlpha software package. (See `https://www.wolframalpha.com/`.) They demonstrate the performance of the asymptotic approximations.

Each of the approximations gets further and further from $n!$ as n tends to infinity. So the quality of the approximations is best judged by considering the percentage error, that is $100 \times \dfrac{\text{approximation} - n!}{n!}$.

S = Stirling, B = Burnside, G = Gosper

Table 1.

n	$n!$	S %error	B %error	G %error
2	2	4.0	1.7	1.3×10^{-1}
5	1.2×10^{2}	1.7	7.6×10^{-1}	2.5×10^{-2}
10	3.6×10^{6}	8.3×10^{-1}	4.0×10^{-1}	6.6×10^{-3}
20	2.4×10^{18}	4.2×10^{-1}	2.0×10^{-1}	1.7×10^{-3}
50	3.0×10^{64}	1.7×10^{-1}	8.3×10^{-2}	2.7×10^{-4}
100	9.3×10^{157}	8.3×10^{-1}	4.1×10^{-2}	6.9×10^{-5}
10^3	4.0×10^{2567}	8.3×10^{-3}	4.2×10^{-3}	6.9×10^{-7}
10^4	2.8×10^{35659}	8.3×10^{-4}	4.2×10^{-4}	6.9×10^{-9}
10^6	8.3×10^{5565708}	8.3×10^{-6}	4.2×10^{-6}	6.9×10^{-13}

M = Mortici, R = Ramanujan,
L4 = (Laplace) Stirling series up to x^{-4}, N = Nemes

TABLE 2.

n	$n!$	M %error	R % error	L4 %error	N %error
2	2	1.0×10^{-2}	3.3×10^{-3}	1.4×10^{-2}	1.7×10^{-3}
5	1.2×10^{2}	5.7×10^{-4}	1.2×10^{-4}	3.5×10^{-4}	2.0×10^{-5}
10	3.6×10^{6}	7.0×10^{-5}	8.6×10^{-6}	7.8×10^{-7}	6.5×10^{-7}
20	2.4×10^{18}	8.7×10^{-6}	5.7×10^{-7}	2.4×10^{-8}	2.0×10^{-8}
50	3.0×10^{64}	5.6×10^{-7}	1.5×10^{-8}	2.5×10^{-10}	2.1×10^{-10}
100	9.3×10^{157}	6.9×10^{-8}	9.5×10^{-10}	7.8×10^{-12}	6.5×10^{12}
10^3	4.0×10^{2567}	6.9×10^{-11}	9.5×10^{-14}	7.8×10^{-17}	6.5×10^{-17}
10^4	2.8×10^{35659}	6.9×10^{-14}	9.5×10^{-18}	7.8×10^{-22}	6.5×10^{-22}
10^6	8.3×10^{5565708}	6.9×10^{-20}	9.5×10^{-26}	7.8×10^{-32}	6.5×10^{-32}

W = Windschitl, HV = Hirschhorn and Villarino, C = Chen, and SAM = Sid Morris

TABLE 3.

n	$n!$	W % error	HV %error	C % error	SAM %error
2	2	1.6×10^{-3}	1.6×10^{-4}	2.2×10^{-4}	2.9×10^{-4}
5	1.2×10^{2}	1.9×10^{-5}	1.5×10^{-6}	5.0×10^{-7}	6.0×10^{-7}
10	3.6×10^{6}	6.1×10^{-7}	3.0×10^{-8}	4.1×10^{-9}	4.9×10^{-9}
20	2.4×10^{18}	1.9×10^{-8}	5.2×10^{-10}	3.2×10^{-11}	3.8×10^{-11}
50	3.0×10^{64}	2.1×10^{-10}	2.3×10^{-12}	5.3×10^{-14}	6.3×10^{-14}
100	9.3×10^{157}	6.2×10^{-12}	3.6×10^{-14}	4.2×10^{-16}	4.9×10^{-16}
10^3	4.0×10^{2567}	6.2×10^{-17}	3.7×10^{-20}	4.17×10^{-23}	4.9×10^{-23}
10^4	2.8×10^{35659}	6.2×10^{-22}	3.7×10^{-26}	4.2×10^{-30}	4.9×10^{-30}
10^6	8.3×10^{5565708}	6.2×10^{-32}	3.7×10^{-38}	4.2×10^{-44}	1.3×10^{-50}

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Thank You