Minimising the number of faces of a class of polytopes

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(joint work with Guillermo Pineda and Julien Ugon)

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In a standard optimisation problem, we have a domain P (possibly a polytope), a reasonable function $g : P \to \mathbb{R}$ (possibly convex), and we wish to find

$$\min_{x \in P} f(x)$$

or perhaps

$$\max_{x \in P} f(x).$$

We will be interested in another optimisation problem; our domain $\mathcal{P}$ will be a collection of polytopes (of the same dimension), and for some natural functions $f : \mathcal{P} \to \mathbb{R}$ we want to find

$$\min_{P \in \mathcal{P}} f(P).$$

Precise upper bounds for the numbers of edges are easy to obtain. If $d = 3$, a polyhedron with v vertices has at most $3v - 6$ edges, with equality iff every face is a triangle. Such maximal examples are easy to construct.

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McMullen (1970) established the corresponding conclusion for k-dimensional faces for all k; this is known as the Upper Bound Theorem.

We are interested in *minimising* the total number of m-dimensional faces (of d-dimensional polytope with a certain number of vertices). We focus mainly on the number of edges, i.e. $m = 1$. Barnette (1973) established a precise lower bound for *simplicial* polytopes, but for general polytopes, lower bounds are not so easy to obtain.

Simple polytopes in higher dimensions

A d-dimensional polytope is *simple* if every vertex has degree d. For any polytope, the sum of the degrees of the vertices is equal to twice the number of edges. So in general there must be at least $\frac{1}{2}dv$ edges, with equality only if there exists a simple polytope with v vertices.

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If there is a simple d-polytope with v vertices, then either
$v = d + 1$ (simplex), $v = 2d$ (prism), $v = 3d - 3$ ($\Delta_{2,d-2}$),
$v = 16$ and $d = 6$, or $v \geq 3d - 1$.
For all d, and all *sufficiently large* v, we have
$\min F_1(v, d) = \frac{1}{2}vd$ if either v or d is even, and
$\min F_1(v, d) = \frac{1}{2}(v + 1)d - 1$ if both v and d are odd.

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More interested in the case when v is small.
Following Grünbaum (1967), we set

$$\phi_m(v,d) = \binom{d+1}{m+1} + \binom{d}{m+1} - \binom{2d+1-v}{m+1}.$$

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(Easy to show that this is false for $v \geq 2d + 1$.)

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Until 2014, no further progress had been made on this problem.

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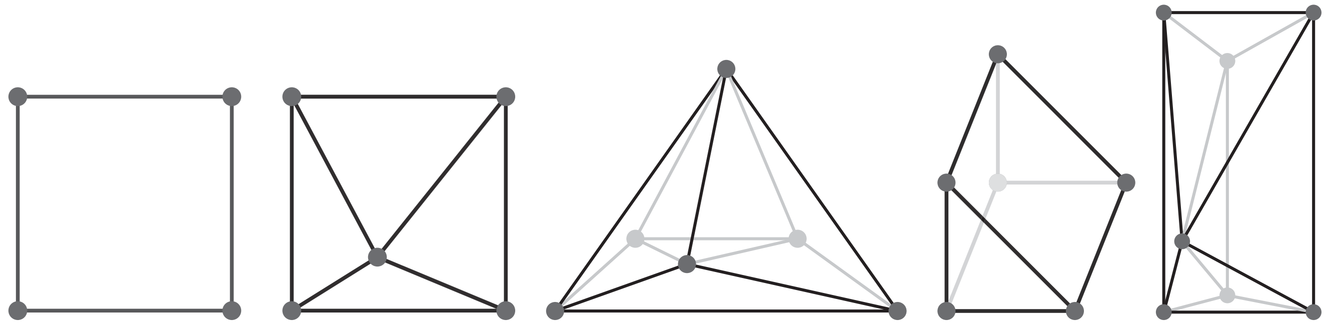
We also obtained precise values for $\phi_1(2d + 1, d)$ and $\phi_1(2d + 2, d)$.

We will not discuss high dimensional faces much, so for simplicity we will mostly write ϕ rather than ϕ_1.

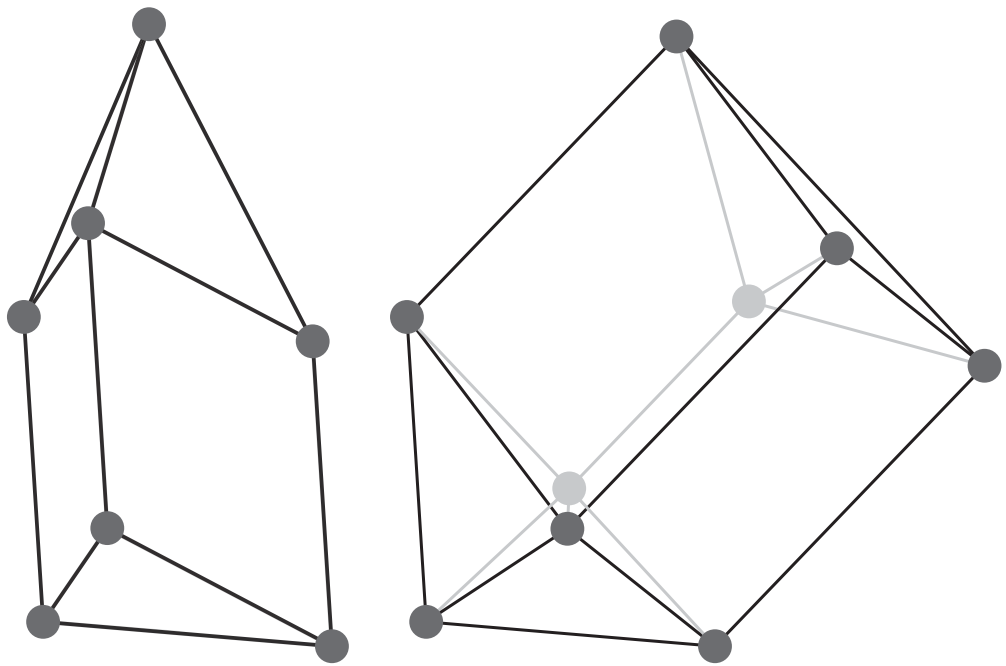
Theorem

Let P be a d-dimensional polytope with $d + k$ vertices, where $0 < k \leq d$.
(i) If P is a $(d - k)$-fold pyramid over the k-dimensional prism based on a simplex, then P has $\phi(d + k, d)$ edges.
(ii) Otherwise P has $> \phi(d + k, d)$ edges.
(iii) Furthermore, P has at least $d - k$ nonsimple vertices, with equality only if P is a $M(k, d - k)$-triplex



(a) $P_2 = M(2,0)$ (b) $M(2,1)$ (c) $M(2,2)$ (d) P_3 (e) $M(3,1)$

FIGURE 1. Triplices



(a) *Pentasm*3 (b) *Pentasm*4

FIGURE 2. Pentasms

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The polytope described in (i) will be denoted $M(k, d-k)$.
The proof depends on the identity

$$\phi(d+k-n, d-1) + nd - \binom{n}{2} = \phi(d+k, d) + (k-n)(n-2).$$

Note that if there are n vertices of a polytope lying outside a given facet, they must belong to at least $nd - \binom{n}{2}$ edges, and the facet must by induction contain sat least $\phi(d+k-n, d-1)$ edges. Observe also that if P had strictly more than $2k$ simple vertices, then it would have strictly more than $\phi(d+k, d)$ edges.

We also proved that Grünbaum's conjecture is true for all faces of sufficiently **high** dimension (more precisely $m \geq 0.62d$). Recently (published online 25/11/21), the problem was solved for all $m \geq 2$ by Xue, with a more elegant argument.

Now, change the question.
Define the *excess degree* of a polytope as

$$2e - dv = \sum_{v \in V} (\deg v - d).$$

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Obviously a polytope is simple iff its excess degree is 0.
Minimising the number of edges is the same as minimising the excess degree.

Theorem

If P is a non-simple d-polytope, then its excess degree is at least $d-2$.

Proof.

Choose a non-simple vertex v in P, and denote by k its excess degree; the conclusion is trivial if $k \geq d-2$. Now the vertex figure of v, i.e. the facet which results when we cut v from P, is a $(d-1)$-polytope with $d+k = d-1+k+1$ vertices, and for $k \leq d-2$ the previous theorem ensures that it has at least $(d-1)-(k+1)$ nonsimple vertices. But every simple neighbour of v in P corresponds to a simple vertex in the vertex figure. Thus at least $d-k-2$ neighbours of v are nonsimple and each of them has excess degree at least 1. This gives P excess degree at least $k+(d-k-2)$. □

We now investigate $\min F_1(2d+1, d)$. We can also calculate $\min F_m(2d+1, d)$ for $m > 1$; the answer depends on some number theory.

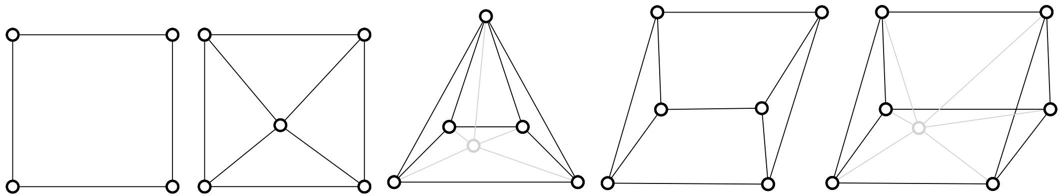
Slicing one corner from the base of a square pyramid yields a polyhedron with 7 vertices and 6 faces, one of them a pentagon. We call this a *pentasm*.

We will use the same name for the higher-dimensional version, obtained by slicing one corner from the quadrilateral base of a $(d-2)$-fold pyramid. It has $2d+1$ vertices and can also be represented as the Minkowski sum of a d-dimensional simplex, and a line segment which lies in the affine span of one 2-face but is not parallel to any edge.

Theorem

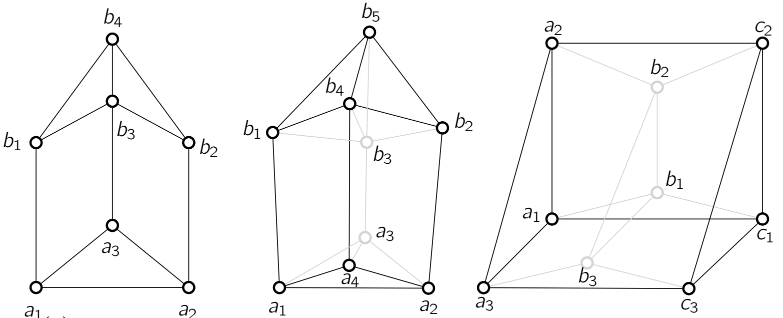
Let P be a d-dimensional polytope with $2d + 1$ vertices.
(i) If P is d-dimensional pentasm, then P has $d^2 + d - 1$ edges.
(ii) Otherwise the numbers of edges is $> d^2 + d - 1$, or P is the sum of two triangles.

This shows that the pentasm is the unique minimiser if $d \geq 5$.
If $d = 4$, the sum of two triangles has 9 vertices, and is the unique minimiser, with only 18 edges.
If $d = 3$, the sum of two triangles can have 7, 8 or 9 vertices; the example with $v = 7$ has 11 edges, the same as the pentasm.



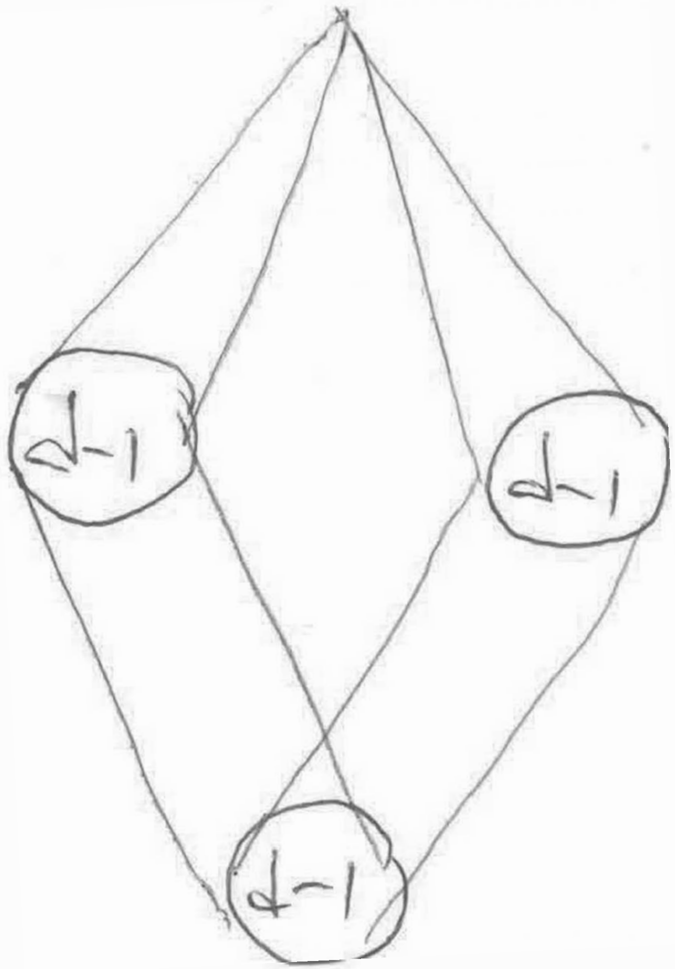
(a) $M_{2,0}$ (b) $M_{2,1}$ (c) $M_{2,2}$ (d) $M_{3,0}$ (e) $M_{3,1}$

Minimizers of the number of edges, for polytopes with no more than *2d* vertices



(a) 3-pentasm (b) 4-pentasm (c) $\Delta_{2,2}$

Minimizers of the number of edges, for polytopes with *2d+1* vertices



$\Sigma(d)$

Slicing one corner from the apex of a square pyramid yields a polyhedron combinatorially equivalent to the cube. Slicing one corner from 3-prism yields a polyhedron combinatorially equivalent to the 5-wedge. Of all the polyhedra with 8 vertices, these are the only two with 12 edges.

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We show that for $d \neq 5$, analogues of these polyhedra minimise the number of edges, amongst polytopes with $2d + 2$ vertices.

Consider first the polytope obtained by slicing one corner from the apex of a $(d-2)$-fold pyramid on a square base. It has $2d+2$ vertices, $(d+1)^2 - 4$ edges and can also be represented as the Minkowski sum of a $(d-3)$-fold pyramid on a square base, and a line segment in the other dimension.

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Consider next a $(d - 3)$-fold pyramid whose base is a 3-prism, then slice one corner off. This example also has $2d + 2$ vertices and $(d + 1)^2 - 4$ edges.

Theorem

Let P be a d-dimensional polytope with $2d+2$ vertices, where $d \geq 6$ or $d = 3$.

(i) If P is one of the two polytopes just described, then P has d^2+2d-3 edges.

(ii) Otherwise the numbers of edges is $> d^2+2d-3$.

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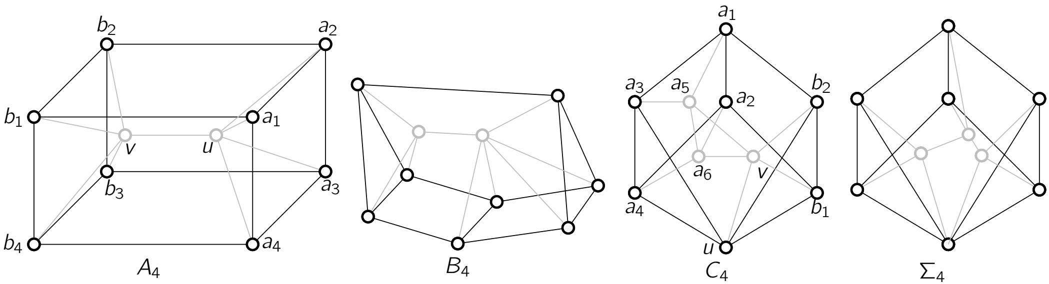
If $d = 4$, there are two more minimising polytopes with 10 vertices and 21 edges.
If $d = 5$, the unique minimiser is the sum of a tetrahedron and triangle; this clearly has 12 vertices and 30 edges; $30 < 32$.
If $d = 7$, there is a third minimising polytope with 16 vertices and 60 edges.

Polytopes with *2d+2* vertices with minimal number of edges

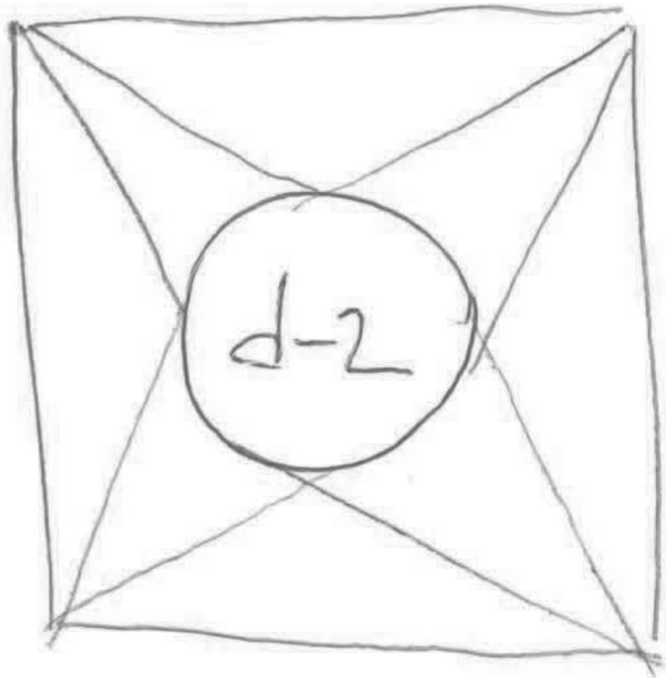


The 4-polytopes with ten vertices and 21 edges.

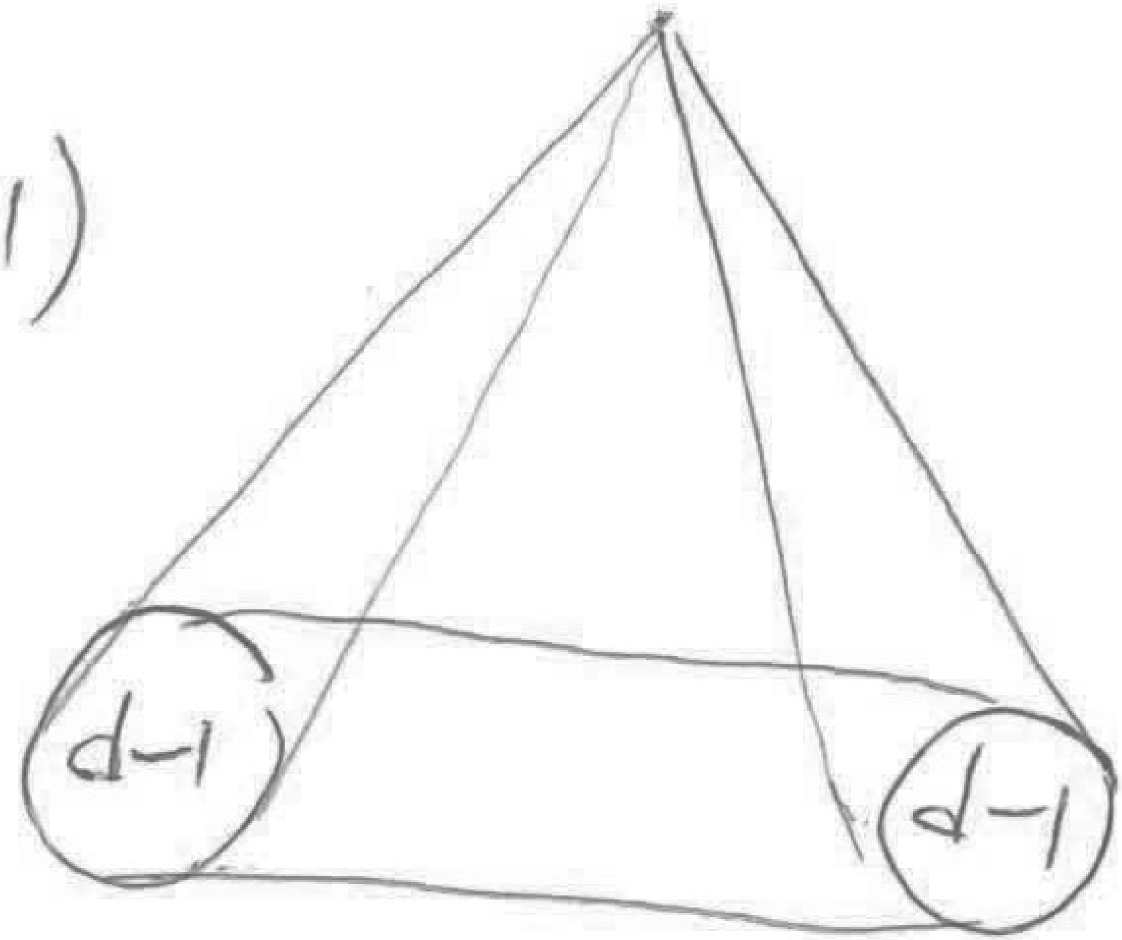
In fact, the set $F_1(d + k, d)$ contains gaps if $k \geq 4$; the number of edges of a non-minimising polytope is at least $\phi(d + k, d) + k - 2$ (expect for $k = 5$).

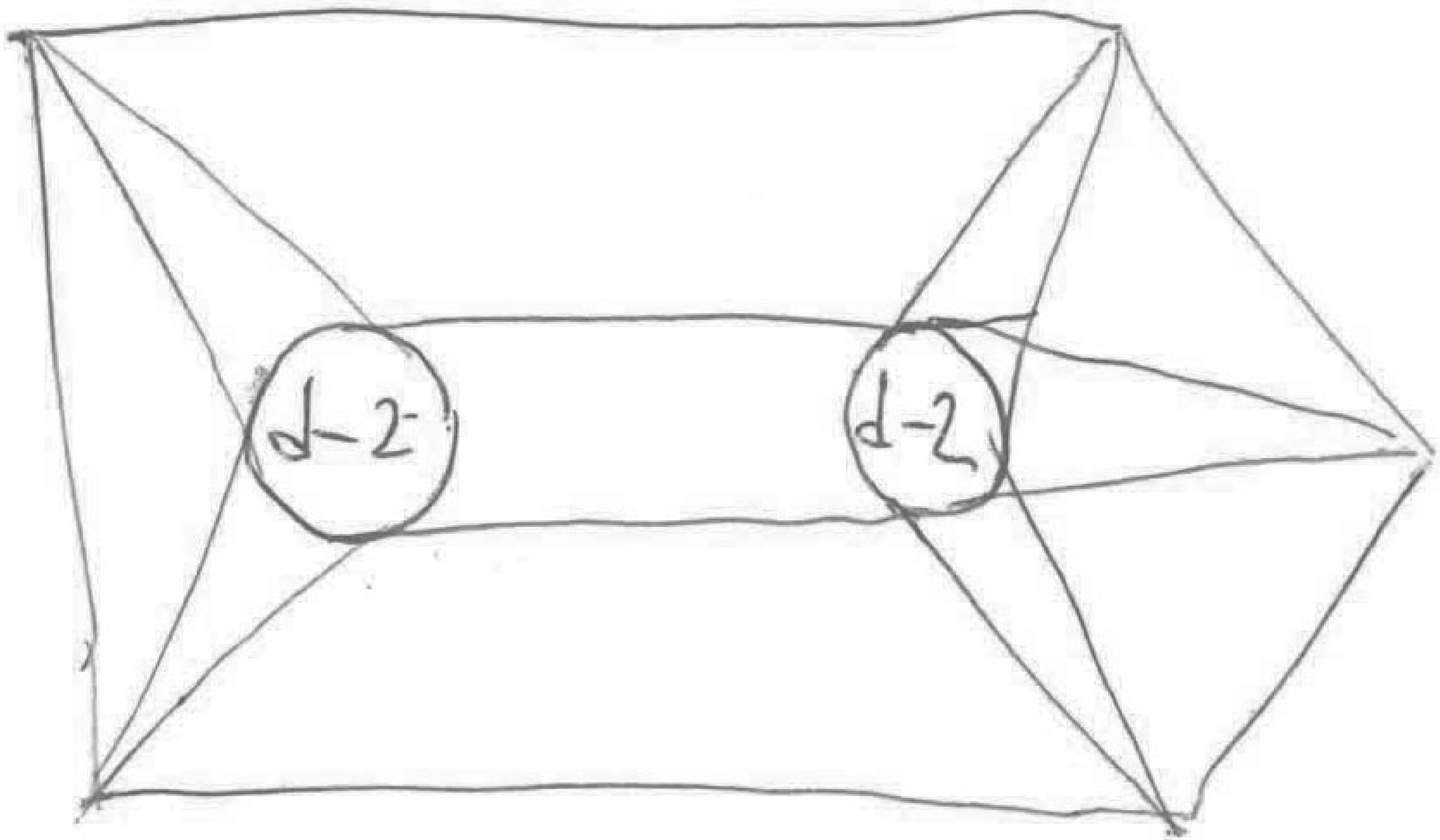
We now show that having low excess degree imposes severe restrictions on the structure of a polytope.
A polytope with excess $d-2$ either has a single vertex with excess $d-2$, or $d-2$ vertices with excess one. In both cases, the nonsimple vertices form a simplex face.
If there is a d-polytope with v vertices with excess $d-2$, then either
$v = d + 2$ (triplex $M(2, d-2)$),
$v = 2d - 1$ (triplex $M(d-1, 1)$),
$v = 2d + 1$ (pentasm),
$v = 3d - 2$ ($C(d), \Sigma(d), N(d), A(4)$),
$v = 3d - 1, d = 4$ (three sporadic examples),
or $v \geq 3d$.

$M(3, d-2)$



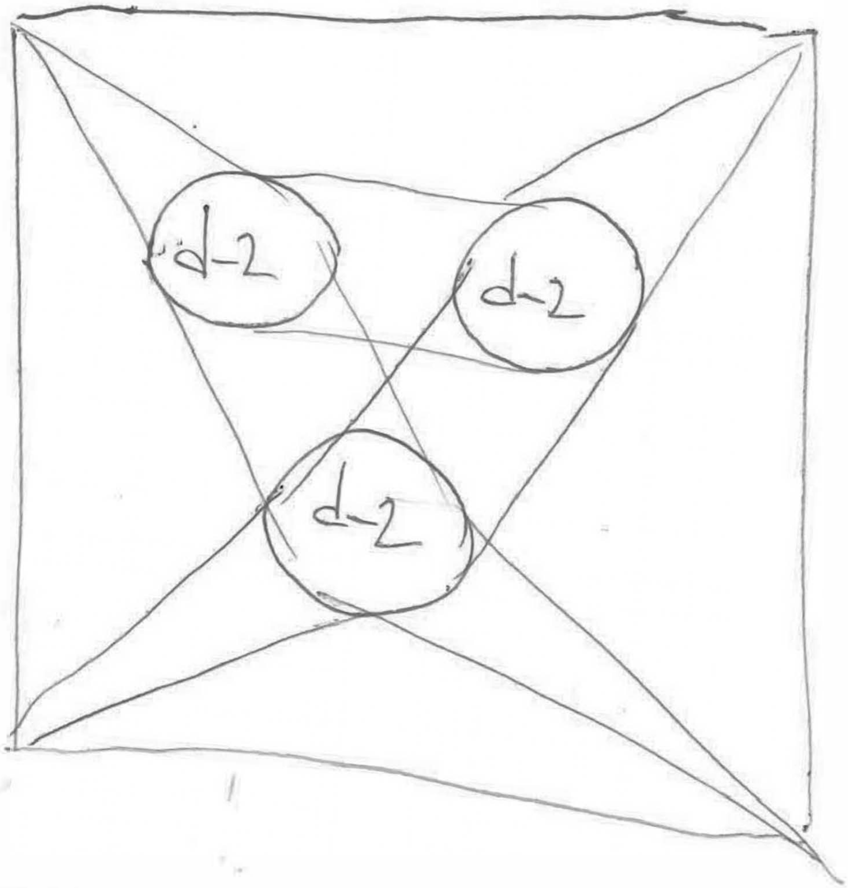
$M(d-1, 1)$

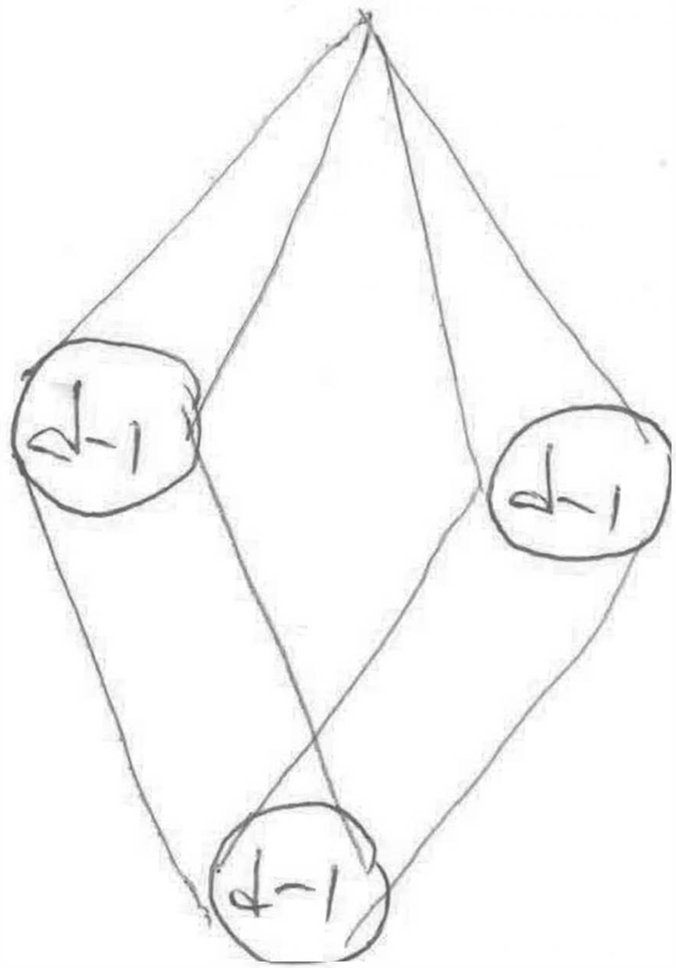




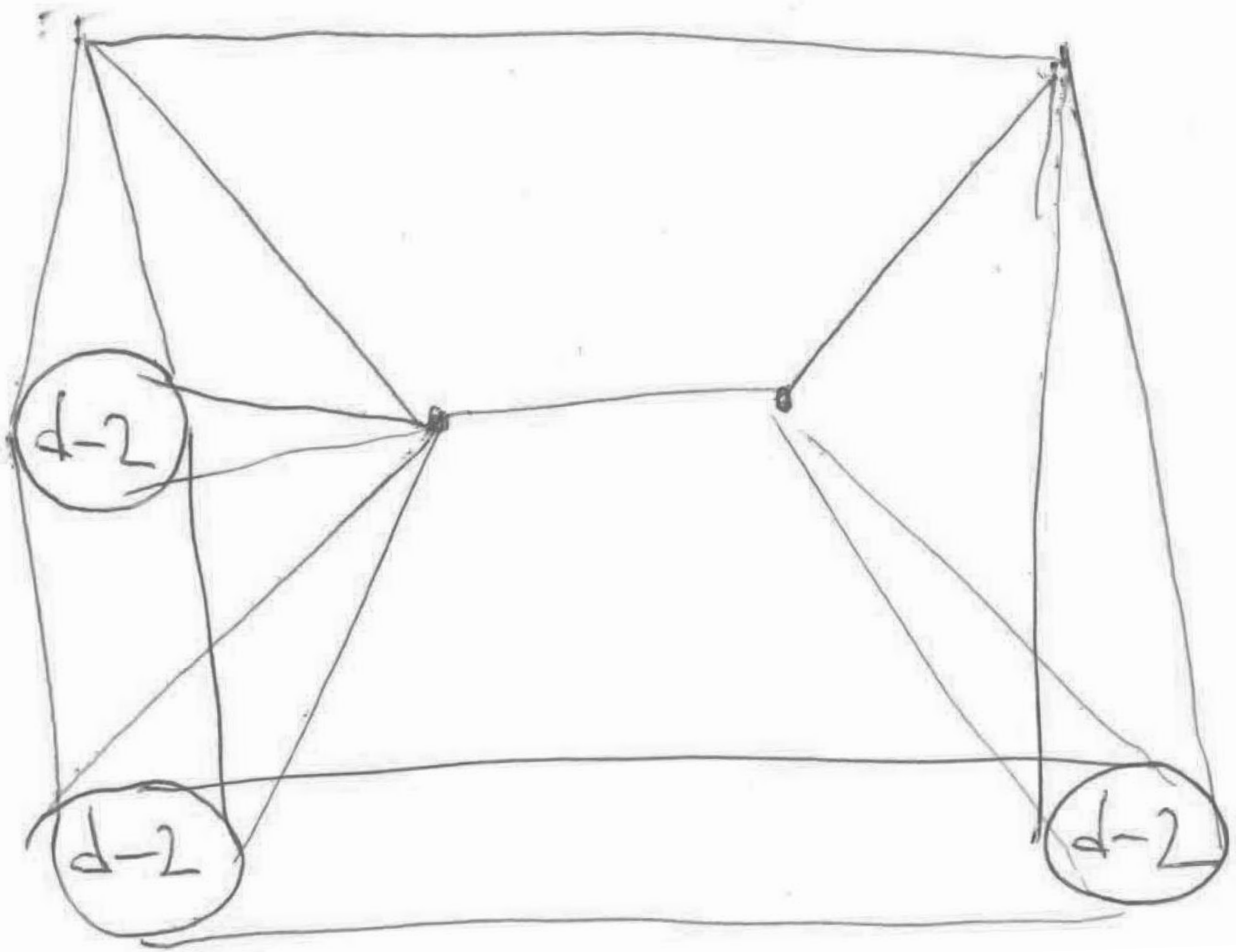
d-pentasm

C(d)





$\Sigma(d)$



N(d)

If a polytope with excess $d-1$, then either $d=3$ (many sporadic examples) or $d=5$, and all nonsimple vertices have the same degree. In the latter case, the nonsimple vertices form a face, which is either a simplex or a square.

So, having excess $d-2$ or $d-1$ limits the possible excess degrees of individual vertices. Examples in dimensions 3 and 4 suggest there is no such structural result for excess d. Not so!

Theorem

Let P be a d-polytope with excess d, where $d \geq 7$ or $d = 5$. Then P has d vertices with excess 1. For $d = 6$, we can still conclude that all nonsimple vertices have the same degree.

For examples in $d = 6$, consider $M(3,3)$ which has three vertices with excess two, $M(4,2)$ which has two vertices with excess three, and the pyramid over $\Delta(2,3)$, which has one vertex with excess six.

Conjecture: If P is a d-polytope with excess d, where $d \geq 7$, then it is formed by gluing two simple polytopes together, along a simplex facet. This would imply that if such a polytope has v vertices, then either $v = d + 2$ (bipyramid over a simplex), $v = 2d + 1$ (capped prism) or $v \geq 3d$.

Towards a classification?

Three dimensional polyhedra are well understood. Catalogues exist of all polyhedra with up to about 20 vertices, and algorithms exist for creating more.

Catalogues exist of all 4-dimensional and 5-dimensional polyhedra with up to 9 vertices.

There are $\frac{d^2}{4}$ polytopes with $d+2$ vertices, and they are very well understood.

Algorithms exist for finding all (combinatorial types of) d-polytopes with $d+3$ vertices. Catalogues exist for $d \leq 6$.

Polytopes with $d+4$ vertices are often considered chaotic. If we restrict our attention to polytopes with low excess/ few edges, things become more tractable.

It seems feasible to characterise all d-polytopes with up to $2d+2$ vertices, and at most two more edges than the minimum possible.

Thank you for
your attention